Homework

Sultan Qaboos University College of Science

Department of Mathematics and Statistics

Calculus II **Spring 2011**

Name:. . . . . Number:. . . . . . . Section:. . . .

Instructions:

(i) Give your complete, mathematically correct and neatly written solution for each question.

(ii) The instructor may grade only one question and it could be any one of them.

(iii) The due date for submitting your solution is Saturday, April 16, 2011 (class time). Late submission will not be accepted under any circumstances.

(iv) The instructor may give you a short quiz out of this homework on Saturday, April 16, 2011.

(v) Copying someone else's homework is considered cheating/plagiarism; and if detected, penalties will be decided according to university regulations. For more details, please see your course syllabus and pages 36 & 37 of SQU Undergraduate Academic Regulations, Third Edition, 2005.

Q1:

(a) Find the value of $a > 0$ such that $\int_0^a e^{-x}dx = \int_a^\infty e^{-x}dx$.

(b) The region bounded by the circle $(x-b)^2 + y^2 = 1, b > 1$ is rotated about the y-axis to form a doughnut. Find the volume of such this doughnut.

Q2:

(a) Is it possible to find an example of a bounded region in the x, y plane that satisfies the following condition: When you revolve the region about the x-axis, you obtain a solid that has a volume equals its surface area. Justify your answer.

(b) Can you give an example of a convergent sequence a_n such that $\sum_{k=0}^\infty a_k$ is divergent? Justify your answer.

(c) Can you give an example of a divergent sequence a_n such that $\sum_{k=0}^\infty a_k$ is convergent? Justify your answer.

Q3:

(a) Evaluate $\int\limits_{-\pi}^{\pi} \cos(mx)\sin(nx)dx$ for all positive integers m and n.

(b) Show that $\int\limits_{-\pi}^{\pi} \cos^2(mx)dx = \int\limits_{-\pi}^{\pi} \sin^2(mx)dx = \pi$, for any positive integer m.

Q4:

(a) Evaluate $\int \frac{dx}{x^3-1}$.

(b) Find a reduction formula for $\int \cot^n(x)\ dx$ where n is a positive integer. Justify your answer.

Q5: Evaluate each one of the following sums:

(a) $\sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+2}\right)$.

(b) $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}}$.

(c) $\sin(x) - \frac{1}{2}\sin^2(x) + \frac{1}{4}\sin^3(x) - \frac{1}{8}\sin^4(x) + \cdots$

Q6:

(a) Are there any values of p for which $\int_0^{\infty} x^p dx$ converges? Justify your answer.

(b) For which values of p does the sum $\sum_0^{\infty} \frac{k^p}{1+k^p}$ converge?

Q7: Consider the sequence a_n that is defined by the recurrence relation $a_{n+1} = \frac{a_n}{(1+a_n)}, n = 0, 1, 2, ...$ and $a_0 > 0$.

(i) If $a_0 = 1$, find a_1, a_2, a_3, a_4 and a_5.

(ii) Show that a_n is always decreasing.

(iii) Show that a_n is always bounded.

Q8:

(a) A student needed to test the improper integral $\int\limits_{-\infty}^{\infty} \sin(x)dx$ for convergence. The student solved the question as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} \sin(x)\, dx &= \int_{-\infty}^{0} \sin(x)\, dx + \int_{0}^{\infty} \sin(x)\, dx && \texttt{(Step 1)}\\
&= \lim_{R\to\infty} \int_{-R}^{0} \sin(x)\, dx + \lim_{R\to\infty} \int_{0}^{R} \sin(x)\, dx && \texttt{(Step 2)}\\
&= \lim_{R\to\infty} [-\cos(x)]_{-R}^{0} + \lim_{R\to\infty} [-\cos(x)]_{0}^{R} && \texttt{(Step 3)}\\
&= \lim_{R\to\infty} (-1 + \cos(-R)) + \lim_{R\to\infty} (-\cos(R) + 1) && \texttt{(Step 4)}\\
&= \lim_{R\to\infty} (-1 + \cos(-R) - \cos(R) + 1) && \texttt{(Step 5)}\\
&= \lim_{R\to\infty} (\cos(-R) - \cos(R)) && \texttt{(Step 6)}\\
&= \lim_{R\to\infty} (\cos(R) - \cos(R)) && \texttt{(Step 7)}\\
&= \lim_{R\to\infty} (0) && \texttt{(Step 8)}\\
&=0.
\end{aligned}$$

Hence, the improper integral $\int_{-\infty}^{\infty} \sin(x)\, dx$ is convergent. `(Step 9)`

Now, think about yourself as the instructor and you want to grade the solution. Is the solution correct or wrong? If wrong, then identify exactly the step that contains the mistake and explain why it is a mistake.

(b) Think about the following statement: "If $f(x) \leq g(x)$ for all $x \geq 0$ and $\int\limits_0^{\infty} g(x)\, dx$ is convergent, then $\int\limits_0^{\infty} f(x)\, dx$ is convergent." If the statement is TRUE then justify your answer, and if it is FALSE then give a counter example.

Good Luck