Final Cal. 2 Fall 2009

Q1.

(a) $$\int_0^1 \frac{e^{-x}}{\sqrt{1-e^{-x}}}\,dx = \lim_{L\to 0^+} \int_L^1 \frac{e^{-x}}{\sqrt{1-e^{-x}}}\,dx \qquad 0<L<1$$

Substitute $u = 1-e^{-x}$ $\quad du = e^{-x}\,dx$

$x = 1 \Rightarrow u = 1-e^{-1}$

$x = L \Rightarrow u = 1-e^{-L}$

$$\lim_{L\to 0^+} \int_{1-e^{-L}}^{1-e^{-1}} \frac{1}{\sqrt{u}}\,du = \lim_{L\to 0^+} 2\left[\sqrt{1-e^{-1}} - \sqrt{1-e^{-L}}\right]$$

$$= 2\sqrt{1-1/e}$$

(b) $$\int_0^{\pi/2} \sin^5\theta \sqrt[3]{\cos\theta}\,d\theta$$

$$= -\int_0^{\pi/2} (1-\cos^2\theta)^2 \sqrt[3]{\cos\theta}\,d(\cos\theta)$$

$$= -\int_0^{\pi/2} \left[\cos^{1/3}\theta - 2\cos^{7/3}\theta + \cos^{13/3}\theta\right] d(\cos\theta)$$

$$= -\left[\frac{3}{4}\cos^{4/3}\theta - \frac{6}{10}\cos^{10/3}\theta + \frac{3}{16}\cos^{16/3}\theta\right]_0^{\pi/2}$$

$$= \frac{3}{4} - \frac{6}{10} + \frac{3}{16} = \frac{27}{80}$$

Q2

(a)

$$I_n = \int \cos^n x \, dx$$

$$= \int \cos^{n-1} x \cos x \, dx$$

Integrating by parts:

$$u = \cos^{n-1} x \qquad dv = \cos x \, dx$$

$$du = (-\sin x)(n-1)\cos^{n-2} x \, dx \qquad v = \sin x$$

$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \left(\cos^{n-2} x - \cos^n x\right) dx$$

We have, $I_n = \cos^{n-1} x \sin x - (n-1) I_n + (n-1)\int \cos^{n-2} x \, dx$

$$\Rightarrow I_n = \int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$$

(b)

$$\int \cos^4 x \, dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x \, dx$$

$n=4$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left[\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right]$$

with $n=2$

$$\therefore \int \cos^4 x \, dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$$

Q3

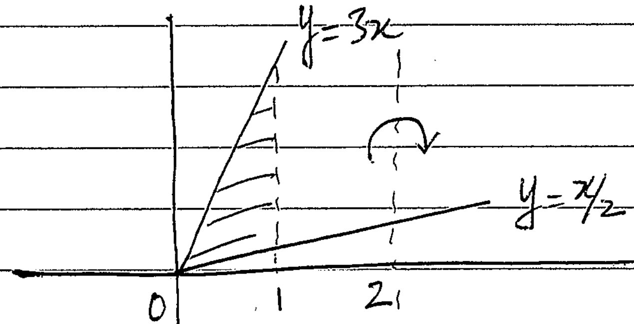
(a)

$$\int_0^1 5\pi x(2-x)\,dx$$

* There are several interpretations of this – All valid interpretations should be credited accordingly –

$$= \int_0^1 2\pi\left(\frac{5x}{2}\right)(2-x)\,dx$$

$$= \int_0^1 2\pi\left(3x-\frac{x}{2}\right)(2-x)\,dx$$



Region enclosed by $y=3x$, $y=x/2$ and $x=1$

Rotation about $x=2$

Note: this is true for any two lines $y=\left(\frac{5}{2}+\alpha\right)x$, $y=\alpha x$ α any real enclosing the region with $x=1$ being the third side.

(b)

$$y=\sqrt{1+5x} \qquad \frac{dy}{dx}=\frac{5}{2\sqrt{1+5x}}$$

$$\text{Surface area} = \int_0^2 2\pi\sqrt{1+5x}\sqrt{1+\frac{25}{4(1+5x)}}\,dx$$

$$= \int_0^2 2\pi\sqrt{1+5x+\frac{25}{4}}\,dx$$

$$= \int_0^2 2\pi\sqrt{\frac{29}{4}+5x}\,dx$$

$$= 2\pi\left[\left(\frac{29}{4}+5x\right)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{5}\right]_0^2$$

$$= \frac{4\pi}{15}\left[\left(\frac{29}{4}+10\right)^{3/2}-\left(\frac{29}{4}\right)^{3/2}\right] \simeq 43.67$$

Q4. LIMIT COMPARISON TEST :

(a) Given two series $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ of positive terms. Let : $L = \lim_{k\to\infty} \frac{a_k}{b_k}$

If $L > 0$ and finite then $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ both converge or both diverge

However, if $L = 0$ then if $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges but if $\sum_{k=1}^{\infty} b_k$ diverges no conclusion can be drawn on $\sum_{k=1}^{\infty} a_k$

And if $L = \infty$ then if $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges but if $\sum_{k=1}^{\infty} b_k$ converges no conclusion can be drawn on $\sum_{k=1}^{\infty} a_k$

(b) $$\sum_{k=1}^{\infty} \frac{2010 + 5^{-k^3}}{k^2(1+e^{-k^2})}$$

For very large values the series behaves like :

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent p-series with $p = 2 > 1$.

Consider, $$\lim_{k\to\infty} \left(\frac{2010+5^{-k^3}}{k^2(1+e^{-k^2})}\right) \Big/ \left(\frac{1}{k^2}\right)$$

$$= \lim_{k\to\infty} \frac{2010+5^{-k^3}}{1+e^{-k^2}} = 2010 > 0$$ and finite

Therefore, by Limit Comparison test,

$$\sum_{k=1}^{\infty} \frac{2010+5^{-k^3}}{k^2(1+e^{-k^2})}$$ Converges

Q5

(a) $1+\frac{1}{e}+\frac{1}{e^2}+\cdots = \sum_{k=0}^{\infty} \frac{1}{e^k} = \sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k$

is a Geometric Series and Convergent since the common ratio, $r=\frac{1}{e}$ which is such that $|r|=|\frac{1}{e}|=\frac{1}{e}<1$.

(b) $a_n = \cos n\pi = (-1)^n$ diverges, since

even seq. $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty}(-1)^{2n} = \lim_{n\to\infty}(1) = 1$

odd seq. $\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} (-1)^{2n-1} = \lim_{n\to\infty}(-1) = -1$

$1 = -1 \Rightarrow \lim a_n$ does not exist.

(c) $\sum_{k=0}^{\infty} (-1)^{k+1} \cos 2k\pi = \sum_{k=0}^{\infty} (-1)^{k+1}(1)$, since $\cos 2k\pi = 1$ for $k \geq 0$

It's an Alternating Series with $a_k = 1$

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (1) = 1 \neq 0$

By k^{th} term test the Series diverges.

(d) $-1+\frac{1}{3}-\frac{1}{5}+\cdots = \sum_{k=1}^{\infty}(-1)^k \frac{1}{2k-1}$

Consider $\sum_{k=1}^{\infty}\left|(-1)^k \frac{1}{2k-1}\right| = \sum_{k=1}^{\infty} \frac{1}{2k-1}$. This behaves like divergent harmonic Series $\sum_{k=1}^{\infty}\frac{1}{k}$ and

$\frac{1}{2k-1} > \frac{1}{2k}$ for all $k \geq 1$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2k-1} > \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{2k-1}$ ~~conv~~ diverges by comparison test as the smaller series diverges.

However, for $\sum_{k=1}^{\infty}(-1)^k \frac{1}{2k-1}$, $\lim_{k\to\infty}\frac{1}{2k-1} = 0$ and $\left\{\frac{1}{2k-1}\right\}$ is decreasing

$\Rightarrow$ by Alternating Series test, it converge. Hence, Conditionally conv.

Q6

(a) $r = \sin\theta + \cos\theta$

$r^2 = r\sin\theta + r\cos\theta$

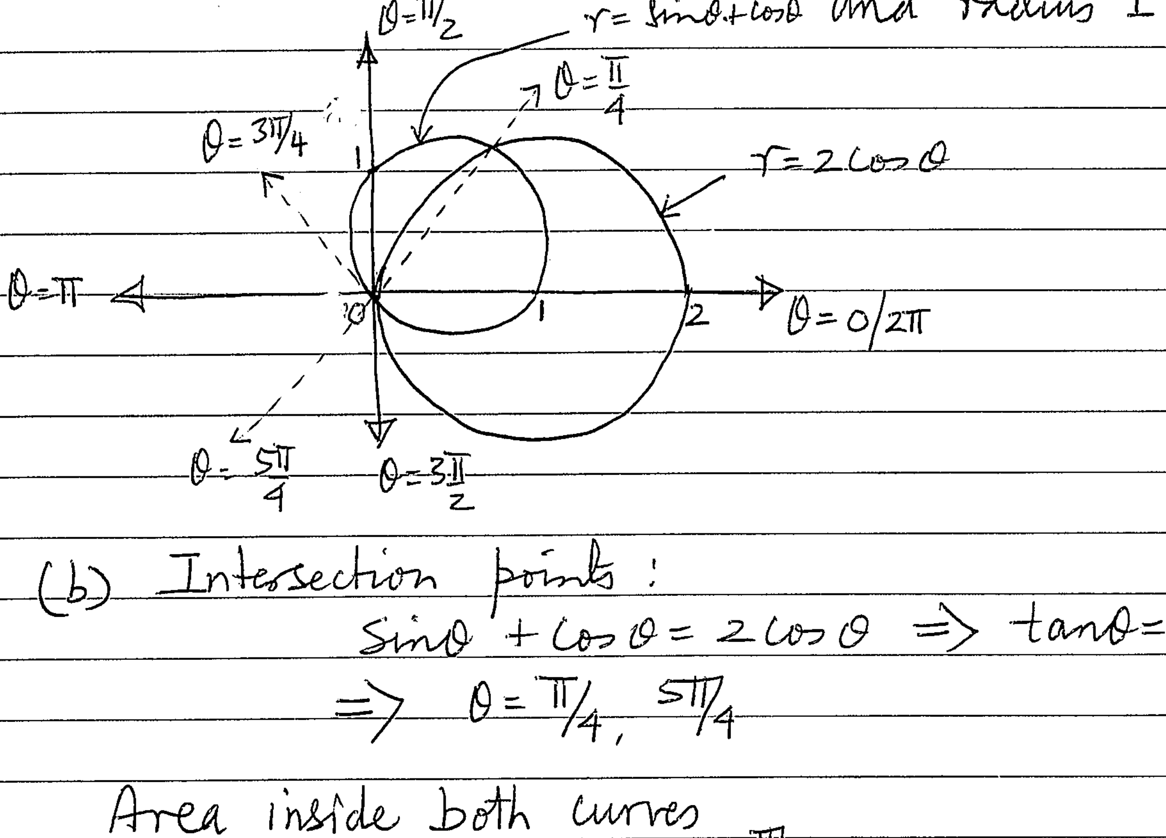
$\Rightarrow x^2+y^2 = y + x$ in Rectangular coords.

$\Rightarrow x^2 - x + y^2 - y = 0$

$\Rightarrow \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

It's a circle with centre $\left(\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$

$r = 2\cos\theta$ is circle with centre at $(1,0)$ and radius 1



(b) Intersection points:

$\sin\theta + \cos\theta = 2\cos\theta \Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \pi/4,\ 5\pi/4$

Area inside both curves

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\theta)^2\,d\theta + \frac{1}{2}\int_0^{\pi/4} (\sin\theta+\cos\theta)^2\,d\theta + \frac{1}{2}\int_{3\pi/4}^{\pi} (\sin\theta+\cos\theta)^2\,d\theta$$

Q7

$$f(x) = \tan^{-1}(x^2)$$

Since, $\frac{d}{dx}(\tan^{-1}x^2) = \frac{2x}{1+x^4}$

$$\Rightarrow \tan^{-1}(x^2) = \int \frac{2x}{1+x^4}\,dx + C$$

Consider the geometric series:

$$1 - x + x^2 - x^3 + \cdots = \sum_{k=0}^{\infty}(-x)^k = \frac{1}{1+x}$$

for $|x|<1$

$$\Rightarrow \frac{2x}{1+x^4} = 2x\sum_{k=0}^{\infty}(-x^4)^k = \sum_{k=0}^{\infty}(-1)^k 2x^{4k+1}$$

Valid for $|-x^4|<1$
$|x|<1$

Therefore, $\int \frac{2x}{1+x^4}\,dx = \int \sum_{k=0}^{\infty}(-1)^k 2x^{4k+1}$

$$\Rightarrow \tan^{-1}x^2 = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{2k+1} + C$$

Valid for $|x|<1$

put $x=0$, $0 = 0 + C \Rightarrow C = 0$

$$\Rightarrow \tan^{-1}x^2 = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{2k+1}$$

(b) for $x=1$, $\tan^{-1}(1) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}$, Convergent

for $x=-1$, $\tan^{-1}(-1) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}$ " "

$\Rightarrow$ Interval of convergence $= [-1,1]$, Radius of conv. $= 1$

(c) for $x=1$, $\tan^{-1}(1) = \frac{\pi}{4} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$