SULTAN QABOOS UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS

Math2108 Summer 2009

Final Exam Time: 165 minutes

Name:. . . . . Solution . . . . Section: . . . . . Number. . . . .

Important Instructions

- Make sure you write your name, number and section number on the exam paper and on the solution booklet.
- Solve all 10 questions. Make sure you show your complete, mathematically correct and neatly written solution.
- You are NOT allowed to share calculators or any other material during the test under any circumstances.
- Cellular phones are NOT allowed to be used as calculators or for any other purpose during the test.
- You should NOT ask the invigilator any questions about the exam.

Q1: Evaluate each one of the following integrals: **(4+5+5+6 points)**

$$(i)\ \int_3^{30} \frac{dx}{(x+2)^{\frac{3}{2}}} \qquad (ii)\ \int_0^2 x\tan^{-1}(x^2)\,dx$$

$$(iii)\ \int e^{5x}\sin^3(e^{5x})\cos(e^{5x})\,dx \qquad (iv)\ \int \frac{(x+2)(x-1)}{(x-2)(x^2+1)}\,dx$$

Solution: (i)

$$\begin{aligned}\int_3^{30} \frac{dx}{(x+2)^{\frac{3}{2}}} &= \left.\frac{(x+2)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1}\right]_3^{30}\\ &= -2(32)^{\frac{-1}{2}} + 2(5)^{\frac{-1}{2}}\\ &= \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{32}}.\end{aligned}$$

(ii)

$$\begin{aligned}\int_0^2 x\tan^{-1}(x^2)\,dx &= \left.\frac{x^2}{2}\tan^{-1}(x^2)\right]_0^2 - \int_0^2 \frac{x^2}{2}\frac{2x}{1+x^4}\,dx\\ &= 2\tan^{-1}4 - 0 - \int_0^2 \frac{x^3}{1+x^4}dx\\ &= 2\tan^{-1}4 - \frac{1}{4}\int_0^2 \frac{4x^3}{1+x^4}dx\\ &= 2\tan^{-1}4 - \frac{1}{4}\ln(1+x^4)]_0^2\\ &= 2\tan^{-1}4 - \frac{1}{4}\ln(17) + \frac{1}{4}\ln(1)\\ &= 2\tan^{-1}4 - \frac{1}{4}\ln(17).\end{aligned}$$

(iii) Let $u = \sin(e^{5x})$, then $du = 5e^{5x}\cos(e^{5x})$. Thus

$$\begin{aligned}\int e^{5x}\sin^3(e^{5x})\cos(e^{5x})\,dx &= \frac{1}{5}\int u^3\,du \\ &= \frac{1}{20}u^4 + c \\ &= \frac{1}{20}\sin^4(e^{5x}) + c.\end{aligned}$$

(iv) We use partial fractions.

$$\begin{aligned}\frac{(x+2)(x-1)}{(x-2)(x^2+1)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+1} \\ (x+2)(x-1) &= A(x^2+1) + (x-2)(Bx+C).\end{aligned}$$

Substitute $x = 2$ to obtain $A = \frac{4}{5}$.
Substitute $x = 0$ to obtain $C = \frac{7}{5}$.
Substitute $x = 1$ to obtain $0 = 2A - (B + C)$, which implies $B = \frac{1}{5}$.
Hence

$$\begin{aligned}\int \frac{(x+2)(x-1)}{(x-2)(x^2+1)}dx &= \frac{4}{5}\int \frac{1}{x-2}dx + \frac{1}{5}\int \frac{x+7}{x^2+1}dx \\ &= \frac{4}{5}\ln|x-2| + \frac{1}{10}\ln(x^2+1) + \frac{7}{5}\tan^{-1}x + c.\end{aligned}$$

Q2: Solve each of (a) and (b). **(5+5 points)**

(a) Find the volume of the solid generated by revolving the region bounded by the curves $y = \ln(x), x = e$ and $y = 0$ about the line $x = -1$.

`Solution:`

`METHOD 1:` Using the method of cylindrical shells.

$$\begin{aligned} V &= \int_1^e 2\pi(x-(-1))\ln(x)\,dx \\ &= 2\pi\int_1^e (x+1)\ln(x)\,dx \\ &= 2\pi\left(\frac{(x+1)^2}{2}\ln x\Big]_1^e - \int_1^e \frac{(x+1)^2}{2}\frac{1}{x}\,dx\right) \\ &= 2\pi\left(\frac{(e+1)^2}{2} - \frac{1}{2}\int_1^e (\frac{x^2}{x}+\frac{2x}{x}+\frac{1}{x})\,dx\right) \\ &= \pi(e+1)^2 - \pi\left(\frac{x^2}{2}+2x+\ln(x)\Big]_1^e\right) \\ &= \frac{1}{2}\pi(e^2+5). \end{aligned}$$

`METHOD 2:` Using the washers method.

$$V = \int_0^1 \pi(e+1)^2\,dy - \int_0^1 \pi(e^y+1)^2\,dy = \frac{\pi}{2}(e^2+5).$$

(b) Determine whether the integral $\int_1^\infty \frac{dx}{(x+1)\ln(x+1)}$ is convergent or divergent.

`Solution:`

$$\int_1^\infty \frac{dx}{(x+1)\ln(x+1)} = \lim_{t\to\infty}\int_1^t \frac{dx}{(x+1)\ln(x+1)}.$$

Let $\ln(x+1) = u$, we obtain $dx = (x+1)du$. Thus, we have

$$\begin{aligned} \lim_{t\to\infty}\int_{\ln(2)}^{\ln(t+1)} \frac{du}{u} &= \lim_{t\to\infty}\left(\ln u\Big]_{\ln 2}^{\ln(t+1)}\right) \\ &= \lim_{t\to\infty}(\ln\ln(t+1)) - \ln\ln(2)) \\ &= \infty. \end{aligned}$$

Hence, the integral is divergent.

Q3: Find the limit (if exists) in each of the following sequences: **(3+4+4 points)**

$$(i) \quad a_n = \frac{(-1)^n n}{3n+1} \qquad (ii) \quad b_n = \sqrt{(4n^2 - n)} - 2n \qquad (iii) \quad c_n = \sqrt{4 - \frac{\sin(3^n)}{5^n}}.$$

`Solution:`

(i) Since $\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$, then $\lim a_n$ oscillates between $\frac{1}{3}$ and $\frac{-1}{3}$. Hence, the limit does not exist.

(ii)

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \left(\sqrt{4n^2 - n} - 2n\right) \frac{(\sqrt{4n^2 - n} + 2n)}{(\sqrt{4n^2 - n} + 2n)} \\
&= \lim_{n\to\infty} \frac{4n^2 - n - 4n^2}{\sqrt{4n^2 - n} + 2n} \\
&= \lim_{n\to\infty} \frac{-n}{\sqrt{4n^2 - n} + 2n} \\
&= \lim_{n\to\infty} \frac{-1}{\sqrt{4 - \frac{1}{n}} + 2} \\
&= \frac{-1}{4}.
\end{aligned}$$

(iii) First, we use squeezing theorem to find $\lim_{n\to\infty} \frac{\sin 3^n}{5^n}$.

$$-1 \leq \sin 3^n \leq 1$$

$$\frac{-1}{5^n} \leq \frac{\sin 3^n}{5^n} \leq \frac{1}{5^n}$$

$$0 \leq \lim_{n\to\infty} \frac{\sin 3^n}{5^n} \leq 0$$

Hence, $\lim_{n\to\infty} \frac{\sin 3^n}{5^n} = 0$, and consequently

$$\lim_{n\to\infty} c_n = \sqrt{4} = 2.$$

Q4: Pick any 2 of the following series and determine whether they converge or diverge. If you solve more than 2, only the first two answered ones will be graded. **(5+5 points)**

$$(i) \sum_{k=1}^{\infty} \frac{3(-1)^k\, k}{\sqrt{k^2+1}} \qquad (ii) \sum_{k=1}^{\infty} \frac{(k^2)(k!)}{(2k)!} \qquad (iii) \sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^{2k}$$

`Solution:`

(i) Since

$$\lim_{k\to\infty} \frac{3(-1)^k\, k}{\sqrt{k^2+1}} = \lim_{k\to\infty} \frac{3(-1)^k}{\sqrt{1+\frac{1}{k}}}$$

does not exist, then it is not zero, and thus, the series in (i) is divergent.

(ii) We use the ratio test.

$$\begin{aligned} \lim_{k\to\infty} \frac{(k+1)^2(k+1)!/(2(k+1))!}{k^2\, k!/(2k)!} &= \lim_{k\to\infty} \frac{(k+1)^2(k+1)}{(2k+2)(2k+1)k^2} \\ &= \lim_{k\to\infty} \frac{(k+1)^2}{2(2k+1)k^2} \\ &= zero. \end{aligned}$$

Since the limit we obtained from the ratio test is less than 1, then the series in (ii) is convergent.

(iii) We use the root test.

$$\lim_{k\to\infty} \left(\left(\frac{k}{2k+1}\right)^{2k}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \left(\frac{k}{2k+1}\right)^{2} = (\frac{1}{2})^2 = \frac{1}{4} < 1.$$

Hence, the root test assures that the series in (iii) is convergent.

Q5: Solve each of (a) and (b) **(4 +4 points)**

(a) Find the sum $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$

(b) Show that $\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k} \cos^k(3x)$ is a geometric series, then find the sum if it exists.

`Solution:`

(a)

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2}{k(k+1)} &= 2\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\ &= 2\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 2\left[(1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \cdots\right] \\ &= 2.\end{aligned}$$

(b)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k} \cos^k(3x) = \sum_{k=0}^{\infty} \left(\frac{(-1)}{3}\cos(3x)\right)^k = \sum_{k=0}^{\infty} r^k,$$

where $r = \frac{-\cos(3x)}{3}$. Hence, it is a geometric series with $a = 1$ and $r = \frac{-1}{3}\cos(3x)$. Since $|r| = \frac{1}{3}|\cos(3x)| \leq \frac{1}{3}$, then the series converges to

$$\frac{a}{1-r} = \frac{1}{1+\frac{1}{3}\cos(3x)}.$$

Q6: Answer each of (a) and (b). **(5 +5 points)**

(a) Consider $f(x) = \ln(x)$.

(i) Construct the Taylor series for $f(x)$ about $c = 1$.

(ii) Find the Taylor series for $g(x) = (x-1)f(x)$ about $c = 1$.

(b) Find the Interval and Radius of Convergence for the power series $\sum_{k=0}^{\infty} \frac{(x-1)^k}{2^k}$.

`Solution:`

(a) The Taylor series of $\ln(x)$ about $c = 1$ is in the form

$$\sum_{k=0}^{\infty} \frac{a_k}{k!}(x-1)^k,$$

where we need to find a_k for all k.

$$\begin{aligned}
a_0 &= f(1) = \ln(1) = 0 \\
a_1 &= f'(1) = \left.\frac{1}{x}\right|_{x=1} = 1 \\
a_2 &= f''(1) = \left.\frac{-1}{x^2}\right|_{x=1} = -1 \\
a_3 &= f'''(1) = \left.\frac{2}{x^3}\right|_{x=1} = 2 \\
a_4 &= f''''(1) = \left.\frac{2(-3)}{x^4}\right|_{x=1} = -3! \\
\vdots &= \vdots
\end{aligned}$$

Hence, $a_0 = 0$ and $a_k = (-1)^{k+1}(k-1)!$ for all $k \geq 1$. Thus,

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k.$$

Now,

$$g(x) = (x-1)f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^{k+1}.$$

(b) We use the ratio test.

$$\lim_{k\to\infty} \left| \frac{(x-1)^{k+1}/2^{k+1}}{(x-1)^k/2^k} \right| = \lim_{k\to\infty} \frac{1}{2}|x-1| = \frac{1}{2}|x-1|.$$

Now,

$$\frac{1}{2}|x-1| < 1 \Leftrightarrow |x-1| < 2 \Leftrightarrow -1 < x < 3.$$

Next, we test the end points. At $x = -1$:

$$\sum_{k=0}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=0}^{\infty} (-1)^k, \quad \text{which is divergent.}$$

At $x = 3 : \sum_{k=0}^{\infty} \frac{(3-1)^k}{2^k} = \sum_{k=0}^{\infty} (1)^k$, which is divergent.

Hence, the interval of convergence is $(-1, 3)$ and the radius of convergence is $\rho = 2$.

Q7: Answer each of (a) and (b).

(a) Given that the Maclaurin series for $f(x) = \frac{x}{1+x^2}$ is

$$x - x^3 + x^5 + ... + (-1)^{k+1}x^{2k-1} + \cdots, \qquad -1 < x < 1.$$

(i) Find the Maclaurin series for $\frac{1}{2}\ln(1+x^2)$. Use sum notation. **(5 points)**

(ii) Evaluate the sum $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(2)^{2k}}$. **(3 points)**

`Solution:` Since

$$\int f(x)dx = \frac{1}{2}\int \frac{2x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + c,$$

then

$$\begin{aligned} \frac{1}{2}\ln(1+x^2) &= \int (x - x^3 + x^5 + ... + (-1)^{k+1}x^{2k-1} + \cdots)dx \\ &= \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} + \cdots + \frac{(-1)^{k+1}x^{2k}}{2k} + \cdots + c \end{aligned}$$

To find c, we substitute $x = 0$ to obtain

$$\frac{1}{2}\ln 1 = 0 = 0 + c.$$

Hence, $c = 0$ and

$$\frac{1}{2}\ln(1+x^2) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{2k}.$$

Next, we obtain (ii) by substituting $x = \frac{1}{2}$ in the power series of $\frac{1}{2}\ln(1+x^2)$. Thus

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(2)^{2k}} = \ln(1 + (\frac{1}{2})^2) = \ln(\frac{5}{4}).$$

(b) Given that **(3 points)**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1.$$

Find the Taylor series for $\frac{1}{1+4x^2}$ about $c = 0$.

`Solution:`

$$\frac{1}{1+4x^2} = \sum_{k=0}^{\infty}(-4x^2)^k = \sum_{k=0}^{\infty}(-1)^k 2^{2k} x^{2k}.$$

Q8: Circle the correct answer. **(2.5 points each)**

(a) One of the following series is convergent:

$$(i)\ \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}} \qquad (ii)\ \sum_{k=1}^{\infty}1 \qquad (iii)\ \sum_{k=1}^{\infty}\frac{(-1)^k}{k} \qquad (iv)\ \sum_{k=1}^{\infty}(-1)^k$$

(b) One of the following series is divergent:

$$(i)\ \sum_{k=1}^{\infty}\left(\frac{-1}{4}\right)^k \qquad (ii)\ \sum_{k=1}^{\infty}\frac{1}{k} \qquad (iii)\ \sum_{k=1}^{\infty}\frac{x^k}{k!} \qquad (iv)\ \sum_{k=1}^{\infty}\frac{\sin(k\pi)}{k\pi}$$

(c) One of the following series is absolutely convergent:

$$(i)\ \sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{k^3+5} \qquad (ii)\ \sum_{k=1}^{\infty}\frac{(-1)^k}{k} \qquad (iii)\ \sum_{k=1}^{\infty}(-\sqrt{3})^k \qquad (iv)\ \sum_{k=1}^{\infty}\frac{\cos(k\pi)}{k}$$

(d) The polar point $(r,\theta)=(-\sqrt{2},\frac{\pi}{4})$ has the rectangular representation $(x,y)=$
(i) $(1,1)$ (ii) $(-1,-1)$ (iii) $(-1,1)$ (iv) $(1,-1)$

Q9: Match each equation with the correct answer in the right column. **(1 point each)**

#	The polar equation	The graph
(i)	$r=-3$	a line
(ii)	$\theta=\frac{3\pi}{5}$	a circle
(iii)	$r=\sin(\theta)$	a cardioid
(iv)	$1=\frac{2\sin(\theta)}{\sin(\theta)+\cos(\theta)}$	a point
(v)	$r=\sin(\theta)+\cos(\theta)$	a parabola

Q10: State whether True or False. No need for justification. **(1 point each)**

(i) A non-power series and its derivative have the same radius of convergence.

(ii) If $\sum_{k=1}^{\infty}a_k$ converges, then $\lim_{k\to\infty}a_k=0$.

(iii) If a series is convergent absolutely, then it is convergent.

(iv) If both $\sum_{k=1}^{\infty}a_k$ and $\sum_{k=1}^{\infty}b_k$ diverge, then $\sum_{k=1}^{\infty}(a_k-b_k)$ diverges.

(v) If a series is convergent, then it is convergent absolutely.

(vi) If a series is conditionally convergent, then it is convergent.

(vii) Each rectangular point (x,y) has a unique (only one) polar representation (r,θ).

Good Luck