CHAPTER 7

Section 7-2

7-1. $$\mathrm{E}\left(\overline{X}_1\right) = E\left(\frac{\sum_{i=1}^{2n} X_i}{2n}\right) = \frac{1}{2n} E\left(\sum_{i=1}^{2n} X_i\right) = \frac{1}{2n}(2n\mu) = \mu$$

$$\mathrm{E}\left(\overline{X}_2\right) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}(n\mu) = \mu,$$ $\overline{X}_1$ and $\overline{X}_2$ are unbiased estimators of μ.

The variances are $\mathrm{V}\left(\overline{X}_1\right) = \frac{\sigma^2}{2\mathrm{n}}$ and $\mathrm{V}\left(\overline{X}_2\right) = \frac{\sigma^2}{\mathrm{n}}$; compare the MSE (variance in this case),

$$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \frac{\sigma^2/2n}{\sigma^2/n} = \frac{n}{2n} = \frac{1}{2}$$

Since both estimators are unbiased, examination of the variances would conclude that $\overline{\mathrm{X}}_1$ is the "better" estimator with the smaller variance.

7-2. $$\mathrm{E}\left(\hat{\Theta}_1\right) = \frac{1}{7}\left[E(X_1) + E(X_2) + \cdots + E(X_7)\right] = \frac{1}{7}(7E(X)) = \frac{1}{7}(7\mu) = \mu$$

$$\mathrm{E}\left(\hat{\Theta}_2\right) = \frac{1}{2}\left[E(2X_1) + E(X_6) + E(X_7)\right] = \frac{1}{2}[2\mu - \mu + \mu] = \mu$$

a) Both $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are unbiased estimates of μ since the expected values of these statistics are equivalent to the true mean, μ.

b) $$\mathrm{V}\left(\hat{\Theta}_1\right) = V\left[\frac{X_1 + X_2 + ... + X_7}{7}\right] = \frac{1}{7^2}\left(V(X_1) + V(X_2) + \cdots + V(X_7)\right) = \frac{1}{49}(7\sigma^2) = \frac{1}{7}\sigma^2$$

$$V(\hat{\Theta}_1) = \frac{\sigma^2}{7}$$

$$\mathrm{V}\left(\hat{\Theta}_2\right) = V\left[\frac{2X_1 - X_6 + X_4}{2}\right] = \frac{1}{2^2}\left(V(2X_1) + V(X_6) + V(X_4)\right) = \frac{1}{4}(4V(X_1) + V(X_6) + V(X_4))$$

$$= \frac{1}{4}\left(4\sigma^2 + \sigma^2 + \sigma^2\right)$$

$$= \frac{1}{4}(6\sigma^2)$$

$$V(\hat{\Theta}_2) = \frac{3\sigma^2}{2}$$

Since both estimators are unbiased, the variances can be compared to decide which is the better estimator. The variance of $\hat{\Theta}_1$ is smaller than that of $\hat{\Theta}_2$, $\hat{\Theta}_1$ is the better estimator.

7-3. Since both $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are unbiased, the variances of the estimators can be examined to determine which is the "better" estimator. The variance of $\hat{\theta}_2$ is smaller than that of $\hat{\theta}_1$ thus $\hat{\theta}_2$ may be the better estimator.

$$\text{Relative Efficiency} = \frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \frac{V(\hat{\Theta}_1)}{V(\hat{\Theta}_2)} = \frac{10}{4} = 2.5$$

7-4. Since both estimators are unbiased:

Relative Efficiency = $\dfrac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \dfrac{V(\hat{\Theta}_1)}{V(\hat{\Theta}_2)} = \dfrac{\sigma^2/7}{3\sigma^2/2} = \dfrac{2}{21}$

7-5. $\dfrac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \dfrac{V(\hat{\Theta}_1)}{V(\hat{\Theta}_2)} = \dfrac{10}{4} = 2.5$

7-6. $E(\hat{\Theta}_1) = \theta \qquad E(\hat{\Theta}_2) = \theta/2$

$Bias = E(\hat{\Theta}_2) - \theta$

$= \dfrac{\theta}{2} - \theta = -\dfrac{\theta}{2}$

$V(\hat{\Theta}_1) = 10 \qquad V(\hat{\Theta}_2) = 4$

For unbiasedness, use $\hat{\Theta}_1$ since it is the only unbiased estimator.

As for minimum variance and efficiency we have:

Relative Efficiency = $\dfrac{(V(\hat{\Theta}_1) + Bias^2)_1}{(V(\hat{\Theta}_2) + Bias^2)_2}$ where, Bias for θ_1 is 0.

Thus,

Relative Efficiency = $\dfrac{(10+0)}{\left(4 + \left(\dfrac{-\theta}{2}\right)^2\right)} = \dfrac{40}{\left(16+\theta^2\right)}$

If the relative efficiency is less than or equal to 1, $\hat{\Theta}_1$ is the better estimator.

Use $\hat{\Theta}_1$, when $\dfrac{40}{(16+\theta^2)} \le 1$

$$40 \le (16+\theta^2)$$
$$24 \le \theta^2$$
$$\theta \le -4.899 \text{ or } \theta \ge 4.899$$

If $-4.899 < \theta < 4.899$ then use $\hat{\Theta}_2$.

For unbiasedness, use $\hat{\Theta}_1$. For efficiency, use $\hat{\Theta}_1$ when $\theta \le -4.899$ or $\theta \ge 4.899$ and use $\hat{\Theta}_2$ when $-4.899 < \theta < 4.899$.

7-7. $E(\hat{\Theta}_1) = \theta$ No bias $V(\hat{\Theta}_1) = 12 = MSE(\hat{\Theta}_1)$

$E(\hat{\Theta}_2) = \theta$ No bias $V(\hat{\Theta}_2) = 10 = MSE(\hat{\Theta}_2)$

$E(\hat{\Theta}_3) \ne \theta$ Bias $MSE(\hat{\Theta}_3) = 6$ [note that this includes (bias^2)]

To compare the three estimators, calculate the relative efficiencies:

$\dfrac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \dfrac{12}{10} = 1.2$, since rel. eff. > 1 use $\hat{\Theta}_2$ as the estimator for θ

$\dfrac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_3)} = \dfrac{12}{6} = 2$, since rel. eff. > 1 use $\hat{\Theta}_3$ as the estimator for θ

$\dfrac{MSE(\hat{\Theta}_2)}{MSE(\hat{\Theta}_3)} = \dfrac{10}{6} = 1.8$, since rel. eff. > 1 use $\hat{\Theta}_3$ as the estimator for θ

Conclusion:

$\hat{\Theta}_3$ is the most efficient estimator with bias, but it is biased. $\hat{\Theta}_2$ is the best "unbiased" estimator.

7-8. $n_1 = 20, n_2 = 10, n_3 = 8$

Show that S^2 is unbiased:

$$E(S^2) = E\left(\frac{20S_1^2 + 10S_2^2 + 8S_3^2}{38}\right)$$
$$= \frac{1}{38}\left(E(20S_1^2) + E(10S_2^2) + E(8S_3^2)\right)$$
$$= \frac{1}{38}\left(20\sigma_1^2 + 10\sigma_2^2 + 8\sigma_3^2\right)$$
$$= \frac{1}{38}\left(38\sigma^2\right)$$
$$= \sigma^2$$

$\therefore$ S^2 is an unbiased estimator of σ^2.

7-9. Show that $\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n}$ is a biased estimator of σ^2 :

a)

$$E\left(\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n}\right)$$
$$= \frac{1}{n}E\left(\sum_{i=1}^{n}\left(X_i - n\overline{X}\right)^2\right)$$
$$= \frac{1}{n}\left(\sum_{i=1}^{n}E\left(X_i^2\right) - nE\left(\overline{X}^2\right)\right)$$
$$= \frac{1}{n}\left(\sum_{i=1}^{n}\left(\mu^2 + \sigma^2\right) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right)$$
$$= \frac{1}{n}\left(n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2\right)$$
$$= \frac{1}{n}((n-1)\sigma^2)$$
$$= \sigma^2 - \frac{\sigma^2}{n}$$

$\therefore$ $\frac{\sum\left(X_i - \overline{X}\right)^2}{n}$ is a biased estimator of σ^2.

b) Bias $= E\left[\frac{\sum\left(X_i^2 - n\overline{X}\right)^2}{n}\right] - \sigma^2 = \sigma^2 - \frac{\sigma^2}{n} - \sigma^2 = -\frac{\sigma^2}{n}$

c) Bias decreases as n increases.

7-10 a) Show that $\overline{X}^2$ is a biased estimator of μ. Using $\mathrm{E}\left(\mathrm{X}^2\right) = \mathrm{V}(\mathrm{X}) + [\mathrm{E}(\mathrm{X})]^2$

$$E\left(\overline{X}^2\right) = \frac{1}{n^2} E\left(\sum_{i=1}^{n} X_i\right)^2$$

$$= \frac{1}{n^2}\left(V\left(\sum_{i=1}^{n} X_i\right) + \left[E\left(\sum_{i=1}^{n} X_i\right)\right]^2\right)$$

$$= \frac{1}{n^2}\left(n\sigma^2 + \left(\sum_{i=1}^{n} \mu\right)^2\right)$$

$$= \frac{1}{n^2}\left(n\sigma^2 + (n\mu)^2\right)$$

$$= \frac{1}{n^2}\left(n\sigma^2 + n^2\mu^2\right)$$

$$E\left(\overline{X}^2\right) = \frac{\sigma^2}{n} + \mu^2$$

$\therefore$ $\overline{X}^2$ is a biased estimator of $\mu.^2$

b) Bias = $E\left(\overline{X}^2\right) - \mu^2 = \frac{\sigma^2}{n} + \mu^2 - \mu^2 = \frac{\sigma^2}{n}$

c) Bias decreases as n increases.

7-11 a.) The average of the 26 observations provided can be used as an estimator of the mean pull force since we know it is unbiased. This value is 75.615 pounds.

b.) The median of the sample can be used as an estimate of the point that divides the population into a "weak" and "strong" half. This estimate is 75.2 pounds.

c.) Our estimate of the population variance is the sample variance or 2.738 square pounds. Similarly, our estimate of the population standard deviation is the sample standard deviation or 1.655 pounds.

d.) The standard error of the mean pull force, estimated from the data provided is 0.325 pounds. This value is the standard deviation, not of the pull force, but of the mean pull force of the population.

e.) Only one connector in the sample has a pull force measurement under 73 pounds. Our point estimate for the proportion requested is then 1/26 = 0.0385

7-12. **Descriptive Statistics**

Variable	N	Mean	Median	TrMean	StDev	SE Mean
Oxide Thickness	24	423.33	424.00	423.36	9.08	1.85

a) The mean oxide thickness, as estimated by Minitab from the sample, is 423.33 Angstroms.
b) Standard deviation for the population can be estimated by the sample standard deviation, or 9.08 Angstroms.
c) The standard error of the mean is 1.85 Angstroms.
d) Our estimate for the median is 424 Angstroms.
e) Seven of the measurements exceed 430 Angstroms, so our estimate of the proportion requested is 7/24 = 0.2917

7.13 a)

$$E(X) = \int_{-1}^{1} x\frac{1}{2}(1+\theta x)dx = \int_{-1}^{1} \frac{1}{2}xdx + \int_{-1}^{1} \frac{\theta}{2}x^2 dx$$
$$= 0 + \frac{\theta}{3} = \frac{\theta}{3}$$

b) Let $\overline{X}$ be the sample average of the observations in the random sample. We know that $E(\overline{X}) = \mu$, the mean of the distribution. However, the mean of the distribution is $\theta/3$, so $\hat{\theta} = 3\overline{X}$ is an unbiased estimator of θ.

7.14 a.) $E(\hat{p}) = E(X/n) = \frac{1}{n}E(X) = \frac{1}{n}np = p$

b.) We know that the variance of $\hat{p}$ is $\frac{p\cdot(1-p)}{n}$ so its standard error must be $\sqrt{\frac{p\cdot(1-p)}{n}}$. To estimate this parameter we would substitute our estimate of p into it.

7.15 a.) $E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu_1 - \mu_2$

b.) $s.e. = \sqrt{V(\overline{X}_1 - \overline{X}_2)} = \sqrt{V(\overline{X}_1) + V(\overline{X}_2) + 2COV(\overline{X}_1, \overline{X}_2)} = \sqrt{\frac{\sigma_1^{\ 2}}{n_1} + \frac{\sigma_2^{\ 2}}{n_2}}$

This standard error could be estimated by using the estimates for the standard deviations of populations 1 and 2.

7-16

$$E(S_p{}^2) = E\left(\frac{(n_1-1)\cdot S_1{}^2 + (n_2-1)\cdot S_2{}^2}{n_1+n_2-2}\right) = \frac{1}{n_1+n_2-2}\left[(n_1-1)E(S_1{}^2) + (n_2-1)\cdot E(S_2{}^2)\right] =$$

$$= \frac{1}{n_1+n_2-2}\left[(n_1-1)\cdot\sigma_1{}^2 + (n_2-1)\cdot\sigma_2{}^2)\right] = \frac{n_1+n_2-2}{n_1+n_2-2}\sigma^2 = \sigma^2$$

7-17 a.) $E(\hat{\mu}) = E(\alpha\overline{X}_1 + (1-\alpha)\overline{X}_2) = \alpha E(\overline{X}_1) + (1-\alpha)E(\overline{X}_2) = \alpha\mu + (1-\alpha)\mu = \mu$

b.)

$$\begin{aligned} s.e.(\hat{\mu}) &= \sqrt{V(\alpha\overline{X}_1 + (1-\alpha)\overline{X}_2)} = \sqrt{\alpha^2 V(\overline{X}_1) + (1-\alpha)^2 V(\overline{X}_2)} \\ &= \sqrt{\alpha^2\frac{\sigma_1{}^2}{n_1} + (1-\alpha)^2\frac{\sigma_2{}^2}{n_2}} = \sqrt{\alpha^2\frac{\sigma_1{}^2}{n_1} + (1-\alpha)^2 a\frac{\sigma_1{}^2}{n_2}} \\ &= \sigma_1\sqrt{\frac{\alpha^2 n_2 + (1-\alpha)^2 an_1}{n_1 n_2}} \end{aligned}$$

c.) The value of alpha that minimizes the standard error is:

$$\alpha = \frac{an_1}{n_2 + an_1}$$

b.) With $a = 4$ and $n_1=2n_2$, the value of alpha to choose is 8/9. The arbitrary value of $\alpha=0.5$ is too small and will result in a larger standard error. With $\alpha=8/9$ the standard error is

$$s.e.(\hat{\mu}) = \sigma_1\sqrt{\frac{(8/9)^2 n_2 + (1/9)^2 8n_2}{2n_2{}^2}} = \frac{0.667\sigma_1}{\sqrt{n_2}}$$

If $\alpha=0.5$ the standard error is

$$s.e.(\hat{\mu}) = \sigma_1\sqrt{\frac{(0.5)^2 n_2 + (0.5)^2 8n_2}{2n_2{}^2}} = \frac{1.0607\sigma_1}{\sqrt{n_2}}$$

7-18 a.) $E(\frac{X_1}{n_1} - \frac{X_2}{n_2}) = \frac{1}{n_1}E(X_1) - \frac{1}{n_2}E(X_2) = \frac{1}{n_1}n_1 p_1 - \frac{1}{n_2}n_2 p_2 = p_1 - p_2 = E(p_1 - p_2)$

b.) $\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$

c.) An estimate of the standard error could be obtained substituting $\frac{X_1}{n_1}$ for p_1 and $\frac{X_2}{n_2}$ for p_2 in the equation shown in (b).

d.) Our estimate of the difference in proportions is 0.01

e.) The estimated standard error is 0.0413

Section 7-3

7-19. $f(x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$ $\quad L(\lambda)=\prod_{i=1}^{n}\dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!}=\dfrac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}$

$$\ln L(\lambda)=-n\lambda\ln e+\sum_{i=1}^{n}x_i\ln\lambda-\sum_{i=1}^{n}\ln x_i!$$

$$\frac{d\ln L(\lambda)}{d\lambda}=-n+\frac{1}{\lambda}\sum_{i=1}^{n}x_i\equiv 0$$

$$-n+\frac{\sum_{i=1}^{n}x_i}{\lambda}=0$$

$$\sum_{i=1}^{n}x_i=n\lambda$$

$$\hat{\lambda}=\frac{\sum_{i=1}^{n}x_i}{n}$$

7-20. $f(x)=\lambda e^{-\lambda(x-\theta)}$ for $x\ge\theta$ $\quad L(\lambda)=\prod_{i=1}^{n}\lambda e^{-\lambda(x-\theta)}=\lambda^{n}e^{-\lambda\sum_{i=1}^{n}(x-\theta)}=\lambda^{n}e^{-\lambda\left(\sum_{i=1}^{n}x-n\theta\right)}$

$$\ln L(\lambda,\theta)=-n\ln\lambda-\lambda\sum_{i=1}^{n}x_i-\lambda n\theta$$

$$\frac{d\ln L(\lambda,\theta)}{d\lambda}=-\frac{n}{\lambda}-\sum_{i=1}^{n}x_i-n\theta\equiv 0$$

$$-\frac{n}{\lambda}=\sum_{i=1}^{n}x_i-n\theta$$

$$\hat{\lambda}=n\Big/\left(\sum_{i=1}^{n}x_i-n\theta\right)$$

$$\hat{\lambda}=\frac{1}{\bar{x}-\theta}$$

Let $\lambda=1$ then $L(\theta)=e^{-\sum_{i=1}^{n}(x_{i-\theta})}$ for $x_{i\geq}0$

$L(\theta)=e^{-\sum_{i=1}^{n}x_{i-n\theta}}=e^{n\theta-n\bar{x}}$

$LnL(\theta)=n\theta-n\bar{x}$

θ cannot be estimated using ML equations since

$\dfrac{dLnL(\theta)}{d(\theta)}=0$. Therefore, θ is estimated using $Min(X_1,X_2,\ldots,X_n)$.

Ln(θ) is maximized at $x_{\min}$ and $\hat{\theta}=x_{\min}$

b.) Example: Consider traffic flow and let the time that has elapsed between one car passing a fixed point and the instant that the next car begins to pass that point be considered time headway. This headway can be modeled by the shifted exponential distribution.

Example in Reliability: Consider a process where failures are of interest. Say that a sample or population is put into operation at x = 0, but no failures will occur until θ period of operation. Failures will occur after the time θ.

7-21. $f(x) = p(1-p)^{x-1}$

$$L(p) = \prod_{i=1}^{n} p(1-p)^{x_i - 1}$$

$$= p^n (1-p)^{\sum_{i=1}^{n} x_i - n}$$

$$\ln L(p) = n \ln p + \left(\sum_{i=1}^{n} x_i - n \right) \ln(1-p)$$

$$\frac{\partial \ln L(p)}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^{n} x_i - n}{1-p} \equiv 0$$

$$0 = \frac{(1-p)n - p\left(\sum_{i=1}^{n} x_i - n \right)}{p(1-p)}$$

$$0 = \frac{n - np - p\sum_{i=1}^{n} x_i + pn}{p(1-p)}$$

$$0 = n - p\sum_{i=1}^{n} x_i$$

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i}$$

7-22. $f(x) = (\theta + 1)x^{\theta}$

$$L(\theta) = \prod_{i=1}^{n} (\theta+1)x_i^{\theta} = (\theta+1)x_1^{\theta} \times (\theta+1)x_2^{\theta} \times \ldots$$

$$= (\theta+1)^n \prod_{i=1}^{n} x_i^{\theta}$$

$$\ln L(\theta) = n\ln(\theta+1) + \theta \ln x_1 + \theta \ln x_2 + \ldots$$

$$= n\ln(\theta+1) + \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta+1} + \sum_{i=1}^{n} \ln x_i = 0$$

$$\frac{n}{\theta+1} = -\sum_{i=1}^{n} \ln x_i$$

$$\hat{\theta} = \frac{n}{-\sum_{i=1}^{n} \ln x_i} - 1$$

7-23. a)

$$L(\beta,\delta)=\prod_{i=1}^{n}\frac{\beta}{\delta}\left(\frac{x_i}{\delta}\right)^{\beta-1}e^{-\left(\frac{x_i}{\delta}\right)^{\beta}}$$

$$=e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\beta}}\prod_{i=1}^{n}\frac{\beta}{\delta}\left(\frac{x_i}{\delta}\right)^{\beta-1}$$

$$\ln L(\beta,\delta)=\sum_{i=1}^{n}\ln\left[\tfrac{\beta}{\delta}\left(\tfrac{x_i}{\delta}\right)^{\beta-1}\right]-\sum\left(\frac{x_i}{\delta}\right)^{\beta}$$

$$=n\ln(\tfrac{\beta}{\delta})+(\beta-1)\sum\ln\left(\tfrac{x_i}{\delta}\right)-\sum\left(\tfrac{x_i}{\delta}\right)^{\beta}$$

b)

$$\frac{\partial\ln L(\beta,\delta)}{\partial\beta}=\frac{n}{\beta}+\sum\ln\left(\tfrac{x_i}{\delta}\right)-\sum\ln\left(\tfrac{x_i}{\delta}\right)\left(\tfrac{x_i}{\delta}\right)^{\beta}$$

$$\frac{\partial\ln L(\beta,\delta)}{\partial\delta}=-\frac{n}{\delta}-(\beta-1)\frac{n}{\delta}+\beta\frac{\sum x_i^{\beta}}{\delta^{\beta+1}}$$

Upon setting $\frac{\partial\ln L(\beta,\delta)}{\partial\delta}$ equal to zero, we obtain

$$\delta^{\beta}n=\sum x_i^{\beta}\quad and\quad \delta=\left[\frac{\sum x_i^{\beta}}{n}\right]^{1/\beta}$$

Upon setting $\frac{\partial\ln L(\beta,\delta)}{\partial\beta}$ equal to zero and substituting for δ, we obtain

$$\frac{n}{\beta}+\sum\ln x_i-n\ln\delta=\frac{1}{\delta^{\beta}}\sum x_i^{\beta}(\ln x_i-\ln\delta)$$

$$\frac{n}{\beta}+\sum\ln x_i-\frac{n}{\beta}\ln\left(\tfrac{\sum x_i^{\beta}}{n}\right)=\frac{n}{\sum x_i^{\beta}}\sum x_i^{\beta}\ln x_i-\frac{n}{\sum x_i^{\beta}}\sum x_i^{\beta}\tfrac{1}{\beta}\ln\left(\tfrac{\sum x_i^{\beta}}{n}\right)$$

and $\frac{1}{\beta}=\left[\frac{\sum x_i^{\beta}\ln x_i}{\sum x_i^{\beta}}+\frac{\sum\ln x_i}{n}\right]$

c) Numerical iteration is required.

7-24 $$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}(\theta+1)x_i^{\theta}=\frac{(\theta+1)}{n}\sum_{i=1}^{n}x_i^{\theta}$$

7-25 $$E(X)=\frac{a-0}{2}=\frac{1}{n}\sum_{i=1}^{n}X_i=\overline{X}$$, therefore: $\hat{a}=2\overline{X}$

The expected value of this estimate is the true parameter so it must be unbiased. This estimate is reasonable in one sense because it is unbiased. However, there are obvious problems. Consider the sample x_1=1, x_2=2 and x_3=10. Now $\overline{x}$ =4.37 and $\hat{a}=2\overline{x}=8.667$. This is an unreasonable estimate of *a*, because clearly $a \geq 10$.

7-26. a) $\hat{a}$ cannot be unbiased since it will always be less than a.

b) bias = $\dfrac{na}{n+1} - \dfrac{a(n+1)}{n+1} = -\dfrac{a}{n+1} \underset{n\to\infty}{\to} 0$.

c) $2\overline{X}$

d) P(Y≤y)=P(X_1, …,X_n ≤ y)=(P(X_1≤y))^n=$\left(\dfrac{y}{a}\right)^n$. Thus, f(y) is as given. Thus,

bias=E(Y)-a = $\dfrac{an}{n+1} - a = -\dfrac{a}{n+1}$.

7-27 For any n>1 $n(n+2) > 3n$ so the variance of $\hat{a}_2$ is less than that of $\hat{a}_1$. It is in this sense that the second estimator is better than the first.

7-28

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i e^{-x_i/\theta}}{\theta^2} \qquad \ln L(\theta) = \sum \ln(x_i) - \sum \frac{x_i}{\theta} - 2n \ln\theta$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{1}{\theta^2}\sum x_i - \frac{2n}{\theta}$$

setting the last equation equal to zero and solving for theta, we find:

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{2n}$$

7-29 a) $$E(X^2) = 2\theta = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \text{ so}$$

$$\hat{\Theta} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$$

b)

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i e^{-x_i^2/2\theta}}{\theta} \qquad \ln L(\theta) = \sum \ln(x_i) - \sum \frac{x_i^2}{2\theta} - n \ln\theta$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{1}{2\theta^2}\sum x_i^2 - \frac{n}{\theta}$$

setting the last equation equal to zero, we obtain the maximum likelihood estimate

$$\hat{\Theta} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$$

which is the same result we obtained in part (a) c)

$$\int_0^a f(x)dx = 0.5 = 1 - e^{-a^2/2\theta}$$

$$a = \sqrt{-2\theta \ln(0.5)}$$

We can estimate the median (*a*) by substituting our estimate for theta into the equation for *a*.

7-30 a) $\int_{-1}^{1} c(1+\theta x)dx = 1 = (cx + c\theta \frac{x^2}{2})\Big|_{-1}^{1} = 2c$

so the constant *c* should equal 0.5

b)

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{\theta}{3}$$

$$\theta = 3 \cdot \frac{1}{n}\sum_{i=1}^{n} X_i$$

c)

$$E(\hat{\theta}) = E\left(3 \cdot \frac{1}{n}\sum_{i=1}^{n} X_i\right) = E(3\overline{X}) = 3E(\overline{X}) = 3\frac{\theta}{3} = \theta$$

d)

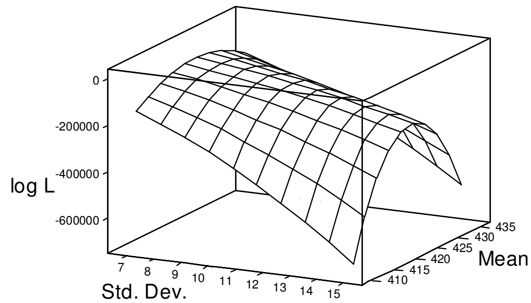
$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2}(1+\theta X_i) \qquad \ln L(\theta) = n\ln(\frac{1}{2}) + \sum_{i=1}^{n} \ln(1+\theta X_i)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{X_i}{(1+\theta X_i)}$$

by inspection, the value of θ that maximizes the likelihood is max (X_i)

7-31 a)Using the results from Example 7-12 we obtain that the estimate of the mean is 423.33 and the estimate of the variance is 82.4464

b)



The function seems to have a ridge and its curvature is not too pronounced. The maximum value for std deviation is at 9.08, although it is difficult to see on the graph.

7-32 When n is increased to 40, the graph will look the same although the curvature will be more pronounced. As n increases it will be easier to determine the where the maximum value for the standard deviation is on the graph.

<u>Section 7-5</u>

7-33. $$P(1.009 \le \overline{X} \le 1.012) = P\left(\frac{1.009-1.01}{0.003/\sqrt{9}} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \frac{1.012-1.01}{0.003/\sqrt{9}}\right)$$

$$= P(-1 \le Z \le 2) = P(Z \le 2) - P(Z \le -1)$$

$$= 0.9772 - 0.1586 = 0.8186$$

7-34. $X_i \sim N(100, 10^2)$ $\quad n = 25$

$$\mu_{\overline{X}} = 100 \quad \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$$

$$P[(100 - 1.8(2)) \le \overline{X} \le (100 + 2)] = P(96.4 \le \overline{X} \le 102)$$

$$= P\left(\frac{96.4-100}{2} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \frac{102-100}{2}\right)$$

$$= P(-1.8 \le Z \le 1) = P(Z \le 1) - P(Z \le -1.8)$$

$$= 0.8413 - 0.0359 = 0.8054$$

7-35. $\mu_{\bar{X}} = 75.5 psi \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{6}} = 1.429$

$$P(\bar{X} \geq 75.75) = P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \geq \frac{75.75-75.5}{1.429}\right)$$

$$= P(Z \geq 0.175) = 1 - P(Z \leq 0.175)$$

$$= 1 - 0.56945 = 0.43055$$

7-36.

n = 6	n = 49
$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{6}}$	$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{49}}$
$= 1.429$	$= 0.5$
$\sigma_{\bar{X}}$ is reduced by 0.929 psi	

7-37. Assuming a normal distribution,

$\mu_{\bar{X}} = 2500 \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{50}{\sqrt{5}} = 22.361$

$$P(2499 \leq \bar{X} \leq 2510) = P\left(\frac{2499-2500}{22.361} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq \frac{2510-2500}{22.361}\right)$$

$$= P(-0.045 \leq Z \leq 0.45) = P(Z \leq 0.45) - P(Z \leq -0.045)$$

$$= 0.6736 - 0.4821 = 0.1915$$

7-38. $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{50}{\sqrt{5}} = 22.361 psi$ = standard error of $\bar{X}$

7-39. $\sigma^2 = 25$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

$$n = \left(\frac{\sigma}{\sigma_{\bar{X}}}\right)^2$$

$$= \left(\frac{5}{1.5}\right)^2$$

$$= 11.11 \sim 12$$

7-40. Let $Y = \overline{X} - 6$

$$\mu_X = \frac{a+b}{2} = \frac{(0+1)}{2} = \tfrac{1}{2}$$

$$\mu_{\overline{X}} = \mu_X$$

$$\sigma_X^2 = \frac{(b-a)^2}{12} = \tfrac{1}{12}$$

$$\sigma_{\overline{X}}^2 = \frac{\sigma^2}{n} = \frac{\frac{1}{12}}{12} = \frac{1}{144}$$

$$\sigma_{\overline{X}} = \tfrac{1}{12}$$

$$\mu_Y = \tfrac{1}{2} - 6 = -5\tfrac{1}{2}$$

$$\sigma_Y^2 = \tfrac{1}{144}$$

$Y = \overline{X} - 6 \sim N(-5\frac{1}{2}, \frac{1}{144})$, approximately, using the central limit theorem.

7-41. $n = 36$

$$\mu_X = \frac{a+b}{2} = \frac{(3+1)}{2} = 2$$

$$\sigma_X = \sqrt{\frac{(b-a+1)^2 - 1}{12}} = \sqrt{\frac{(3-1+1)^2 - 1}{12}} = \sqrt{\frac{8}{12}} = \sqrt{\frac{2}{3}}$$

$$\mu_{\overline{X}} = 2, \sigma_{\overline{X}} = \frac{\sqrt{2/3}}{\sqrt{36}} = \frac{\sqrt{2/3}}{6}$$

$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

Using the central limit theorem:

$$P(2.1 < \overline{X} < 2.5) = P\left(\frac{2.1-2}{\frac{\sqrt{2/3}}{6}} < Z < \frac{2.5-2}{\frac{\sqrt{2/3}}{6}}\right)$$

$$= P(0.7348 < Z < 3.6742)$$

$$= P(Z < 3.6742) - P(Z < 0.7348)$$

$$= 1 - 0.7688 = 0.2312$$

7-42.

$\mu_X = 8.2$ minutes	n = 49
$\sigma_X = 1.5$ minutes	$\sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{1.5}{\sqrt{49}} = 0.2143$
	$\mu_{\overline{X}} = \mu_X = 8.2$ mins

Using the central limit theorem, $\overline{X}$ is approximately normally distributed.

a) $P(\overline{X} < 10) = P(Z < \dfrac{10 - 8.2}{0.2143}) = P(Z < 8.4) = 1$

b) $P(5 < \overline{X} < 10) = P(\frac{5-8.2}{0.2143} < Z < \frac{10-8.2}{0.2143})$

$$= P(Z < 8.4) - P(Z < -14.932) = 1 - 0 = 1$$

c) $P(\overline{X} < 6) = P(Z < \dfrac{6 - 8.2}{0.2143}) = P(Z < -10.27) = 0$

7-43.

$n_1 = 16$	$n_2 = 9$	$\overline{X}_1 - \overline{X}_2 \sim N(\mu_{\overline{X}_1} - \mu_{\overline{X}_2}, \sigma^2_{\overline{X}_1} + \sigma^2_{\overline{X}_2})$
$\mu_1 = 75$	$\mu_2 = 70$	
$\sigma_1 = 8$	$\sigma_2 = 12$	$\sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$
		$\sim N(75-70, \frac{8^2}{16} + \frac{12^2}{9})$
		$\sim N(5,20)$

a) $P(\overline{X}_1 - \overline{X}_2 > 4)$

$$P(Z > \tfrac{4-5}{\sqrt{20}}) = P(Z > -0.2236) = 1 - P(Z \le -0.2236)$$
$$= 1 - 0.4115 = 0.5885$$

b) $P(3.5 \le \overline{X}_1 - \overline{X}_2 \le 5.5)$

$$P(\tfrac{3.5-5}{\sqrt{20}} \le Z \le \tfrac{5.5-5}{\sqrt{20}}) = P(Z \le 0.1118) - P(Z \le -0.3354)$$
$$= 0.5445 - 0.3687 = 0.1759$$

7-44. If $\mu_B = \mu_A$, then $\overline{X}_B - \overline{X}_A$ is approximately normal with mean 0 and variance $\frac{\sigma_B^2}{25} + \frac{\sigma_A^2}{25} = 20.48$.

Then, $P(\overline{X}_B - \overline{X}_A > 3.5) = P(Z > \frac{3.5-0}{\sqrt{20.48}}) = P(Z > 0.773) = 0.2196$

The probability that $\overline{X}_B$ exceeds $\overline{X}_A$ by 3.5 or more is not that unusual when μ_B *and* μ_A are equal. Therefore, there is not strong evidence that μ_B is greater than μ_A.

7-45. Assume approximate normal distributions.

$$(\overline{X}_{high} - \overline{X}_{low}) \sim N(60-55, \tfrac{4^2}{16} + \tfrac{4^2}{16})$$
$$\sim N(5,2)$$
$$P(\overline{X}_{high} - \overline{X}_{low} \ge 2) = P(Z \ge \tfrac{2-5}{\sqrt{2}}) = 1 - P(Z \le -2.12) = 1 - 0.0170 = 0.983$$

Supplemental Exercises

7-46. $$f(x_1, x_2, x_3, x_4, x_5) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\sum_{i=1}^{5} \frac{(x_i-\mu)^2}{2\sigma^2}}$$

7-47. $$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \quad for \quad x_1 > 0, x_2 > 0, ..., x_n > 0$$

7-48. $$f(x_1, x_2, x_3, x_4) = 1 \quad for \quad 0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1, 0 \le x_4 \le 1$$

7-49.

$$\overline{X}_1 - \overline{X}_2 \sim N(100-105, \frac{1.5^2}{25} + \frac{2^2}{30})$$
$$\sim N(-5, 0.2233)$$

7-50. $\sigma_{\overline{X}_1-\overline{X}_2} = \sqrt{0.2233} = 0.4726$

7-51. $X \sim N(50,144)$

$$P(47 \le \overline{X} \le 53) = P\left(\frac{47-50}{12/\sqrt{36}} \le Z \le \frac{53-50}{12/\sqrt{36}}\right)$$

$$= P(-1.5 \le Z \le 1.5)$$

$$= P(Z \le 1.5) - P(Z \le -1.5)$$

$$= 0.9332 - 0.0668 = 0.8664$$

7-52. No, because Central Limit Theorem states that with large samples ($n \ge 30$), $\overline{X}$ is approximately normally distributed.

7-53. Assume $\overline{X}$ is approximately normally distributed.

$$P(\overline{X} > 4985) = 1 - P(\overline{X} \le 4985) = 1 - P(Z \le \tfrac{4985-5500}{100/\sqrt{9}})$$

$$= 1 - P(Z \le -15.45) = 1 - 0 = 1$$

7-54. $$t = \frac{\overline{X}-\mu}{s/\sqrt{n}} = \frac{52-50}{\sqrt{2/16}} = 5.6569$$

$t_{.05,15} = 1.753$. Since 5.33 >> $t_{.05,15}$, the results are <u>very unusual</u>.

7-55. $P(\overline{X} \le 37) = P(Z \le -5.36) = 0$

7-56. Binomial with p equal to the proportion of defective chips and n = 100.

7-57. $$E(a\overline{X}_1 + (1-a)\overline{X}_2 = a\mu + (1-a)\mu = \mu$$

$$\begin{aligned}
V(\overline{X}) &= V[a\overline{X}_1 + (1-a)\overline{X}_2] \\
&= a^2V(\overline{X}_1) + (1-a)^2V(\overline{X}_2) \\
&= a^2(\tfrac{\sigma^2}{n_1}) + (1-2a+a^2)(\tfrac{\sigma^2}{n_2}) \\
&= \frac{a^2\sigma^2}{n_1} + \frac{\sigma^2}{n_2} - \frac{2a\sigma^2}{n_2} + \frac{a^2\sigma^2}{n_2} \\
&= (n_2a^2 + n_1 - 2n_1a + n_1a^2)(\frac{\sigma^2}{n_1n_2})
\end{aligned}$$

$$\frac{\partial V(\overline{X})}{\partial a} = (\frac{\sigma^2}{n_1n_2})(2n_2a - 2n_1 + 2n_1a) \equiv 0$$

$$0 = 2n_2a - 2n_1 + 2n_1a$$

$$2a(n_2+n_1) = 2n_1$$

$$a(n_2+n_1) = n_1$$

$$a = \frac{n_1}{n_2+n_1}$$

7-58

$$L(\theta) = \left(\frac{1}{2\theta^3}\right)^n e^{\sum_{i=1}^{n} \frac{-x_i}{\theta}} \prod_{i=1}^{n} x_i^{\,2}$$

$$\ln L(\theta) = n \ln\left(\frac{1}{2\theta^3}\right) + 2\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \frac{x_i}{\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{-3n}{\theta} + \sum_{i=1}^{n} \frac{x_i}{\theta^2}$$

Making the last equation equal to zero and solving for theta, we obtain:

$$\hat{\Theta} = \frac{\sum_{i=1}^{n} x_i}{3n}$$

as the maximum likelihood estimate.

7-59

$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{\,\theta-1}$$

$$\ln L(\theta) = n \ln \theta + (\theta - 1)\sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i)$$

making the last equation equal to zero and solving for theta, we obtain the maximum likelihood estimate.

$$\hat{\Theta} = \frac{-n}{\sum_{i=1}^{n} \ln(x_i)}$$

7-60

$$L(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} x_i^{\frac{1-\theta}{\theta}}$$

$$\ln L(\theta) = -n \ln \theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \ln(x_i)$$

making the last equation equal to zero and solving for the parameter of interest, we obtain the maximum likelihood estimate.

$$\hat{\Theta} = -\frac{1}{n} \sum_{i=1}^{n} \ln(x_i)$$

$$E(\hat{\theta}) = E\left[-\frac{1}{n} \sum_{i=1}^{n} \ln(x_i)\right] = \frac{1}{n} E\left[-\sum_{i=1}^{n} \ln(x_i)\right] = -\frac{1}{n} \sum_{i=1}^{n} E[\ln(x_i)]$$
$$= \frac{1}{n} \sum_{i=1}^{n} \theta = \frac{n\theta}{n} = \theta$$

$E(\ln(X_i)) = \int_0^1 (\ln x) x^{\frac{1-\theta}{\theta}} dx$ let $u = \ln x$ and $dv = x^{\frac{1-\theta}{\theta}} dx$

then, $E(\ln(X)) = -\theta \int_0^1 x^{\frac{1-\theta}{\theta}} dx = -\theta$

Mind-Expanding Exercises

7-61. $P(X_1 = 0, X_2 = 0) = \dfrac{M(M-1)}{N(N-1)}$

$P(X_1 = 0, X_2 = 1) = \dfrac{M(N-M)}{N(N-1)}$

$P(X_1 = 1, X_2 = 0) = \dfrac{(N-M)M}{N(N-1)}$

$P(X_1 = 1, X_2 = 1) = \dfrac{(N-M)(N-M-1)}{N(N-1)}$

$P(X_1 = 0) = M/N$

$P(X_1 = 1) = \dfrac{N-M}{N}$

$$P(X_2 = 0) = P(X_2 = 0 \mid X_1 = 0)P(X_1 = 0) + P(X_2 = 0 \mid X_1 = 1)P(X_1 = 1)$$
$$= \frac{M-1}{N-1} \times \frac{M}{N} + \frac{M}{N-1} \times \frac{N-M}{N} = \frac{M}{N}$$
$$P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 0)P(X_1 = 0) + P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1)$$
$$= \frac{N-M}{N-1} \times \frac{M}{N} + \frac{N-M-1}{N-1} \times \frac{N-M}{N} = \frac{N-M}{N}$$

Because $P(X_2 = 0 \mid X_1 = 0) = \frac{M-1}{N-1}$ is not equal to $P(X_2 = 0) = \dfrac{M}{N}$, X_1 *and* X_2 are not independent.

7-62 a)

$$c_n = \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)\sqrt{2/(n-1)}}$$

b.) When $n = 10$, $c_n = 1.0281$. When $n = 25$, $c_n = 1.0105$. So S is a pretty good estimator for the standard deviation even when relatively small sample sizes are used.

7-63 (a) $Z_i = Y_i - X_i$; so Z_i is N(0, $2\sigma^2$). Let $\sigma^{*2} = 2\sigma^2$. The likelihood function is

$$L(\sigma^{*2}) = \prod_{i=1}^{n} \frac{1}{\sigma^* \sqrt{2\pi}} e^{-\frac{1}{2\sigma^{*2}}(z_i^2)}$$

$$= \frac{1}{(\sigma^{*2}\, 2\pi)^{n/2}} e^{-\frac{1}{2\sigma^{*2}}\sum_{i=1}^{n} z_i^2}$$

The log-likelihood is $\ln[L(\sigma^{*2})] = -\frac{n}{2}(2\pi\sigma^{*2}) - \frac{1}{2\sigma^{*2}}\sum_{i=1}^{n} z_i^2$

Finding the maximum likelihood estimator:

$$\frac{d\ln[L(\sigma^{*2})]}{d\sigma^*} = -\frac{n}{2\sigma^{*2}} + \frac{1}{2\sigma^{*4}}\sum_{i=1}^{n} z_i^2 = 0$$

$$n\sigma^{*2} = \sum_{i=1}^{n} z_i^2$$

$$\hat{\sigma}^{*2} = \frac{1}{n}\sum_{i=1}^{n} z_i^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - x_i)^2$$

But $\sigma^{*2} = 2\sigma^2$, so the MLE is

$$2\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - x_i)^2$$

$$\hat{\sigma}^2 = \frac{1}{2n}\sum_{i=1}^{n}(y_i - x_i)^2$$

b)

$$E(\hat{\sigma}^2) = E\left[\frac{1}{2n}\sum_{i=1}^{n}(Y_i - X_i)^2\right]$$

$$= \frac{1}{2n}\sum_{i=1}^{n} E(Y_i - X_i)^2$$

$$= \frac{1}{2n}\sum_{i=1}^{n} E(Y_i^2 - 2Y_iX_i + X_i^2)$$

$$= \frac{1}{2n}\sum_{i=1}^{n}[E(Y_i^2) - E(2Y_iX_i) + E(X_i^2)]$$

$$= \frac{1}{2n}\sum_{i=1}^{n}[\sigma^2 - 0 + \sigma^2]$$

$$= \frac{2n\sigma^2}{2n}$$

$$= \sigma^2$$

So the estimator is unbiased.

7-64. $P\left(|\overline{X}-\mu|\geq \frac{c\sigma}{\sqrt{n}}\right)\leq \frac{1}{c^2}$ from Chebyshev's inequality.

Then, $P\left(|\overline{X}-\mu|< \frac{c\sigma}{\sqrt{n}}\right)\geq 1-\frac{1}{c^2}$. Given an ε, n and c can be chosen sufficiently large that the last probability is near 1 and $\frac{c\,\sigma}{\sqrt{n}}$ is equal to ε.

7-65 $P(X_{(n)}\leq t)=P(X_i\leq t \text{ for } i=1,...,n)=[F(t)]^n$

$P(X_{(1)}>t)=P(X_i>t \text{ for } i=1,...,n)=[1-F(t)]^n$

Then, $P(X_{(1)}\leq t)=1-[1-F(t)]^n$

$f_{X_{(1)}}(t)=\frac{\partial}{\partial t}F_{X_{(1)}}(t)=n[1-F(t)]^{n-1}f(t)$

$f_{X_{(n)}}(t)=\frac{\partial}{\partial t}F_{X_{(n)}}(t)=n[F(t)]^{n-1}f(t)$

7-66 $P(X_{(1)}=0)=F_{X_{(1)}}(0)=1-[1-F(0)]^n=1-p^n$ because F(0) = 1 - p.

$P(X_{(n)}=1)=1-F_{X_{(n)}}(0)=1-[F(0)]^n=1-(1-p)^n$

7-67. $P(X\leq t)=F(t)=\Phi\left[\frac{t-\mu}{\sigma}\right]$. From Exercise 7-65,

$f_{X_{(1)}}(t)=n\left\{1-\Phi\left[\frac{t-\mu}{\sigma}\right]\right\}^{n-1}\frac{1}{\sqrt{2\pi}\sigma}\;e^{-\frac{(t-\mu)^2}{2\sigma^2}}$

$f_{X_{(n)}}(t)=n\left\{\Phi\left[\frac{t-\mu}{\sigma}\right]\right\}^{n-1}\frac{1}{\sqrt{2\pi}\sigma}\;e^{-\frac{(t-\mu)^2}{2\sigma^2}}$

7-68. $P(X\leq t)=1-e^{-\lambda t}$. From Exercise 7-65,

$F_{X_{(1)}}(t)=1-e^{-n\lambda t}$	$f_{X_{(1)}}(t)=n\lambda e^{-n\lambda t}$
$F_{X_{(n)}}(t)=[1-e^{-\lambda t}]^n$	$f_{X_{(n)}}(t)=n[1-e^{-\lambda t}]^{n-1}\lambda e^{-\lambda t}$

7-69. $P(F(X_{(n)}) \le t) = P(X_{(n)} \le F^{-1}(t)) = t^n$ from Exercise 7-65 for $0 \le t \le 1$.

If $Y = F(X_{(n)})$, then $f_Y(y) = ny^{n-1}, 0 \le y \le 1$. Then, $E(Y) = \int_0^1 ny^n dy = \frac{n}{n+1}$

$P(F(X_{(1)}) > t) = P(X_{(1)} < F^{-1}(t)) = 1 - (1-t)^n$ from Exercise 7-65 for $0 \le t \le 1$.

If $Y = F(X_{(1)})$, then $f_Y(y) = n(1-t)^{n-1}, 0 \le y \le 1$.

Then, $E(Y) = \int_0^1 yn(1-y)^{n-1} dy = \frac{1}{n+1}$ where integration by parts is used. Therefore,

$E[F(X_{(n)})] = \frac{n}{n+1}$ *and* $E[F(X_{(1)})] = \frac{1}{n+1}$

7-70.
$$\begin{aligned} E(V) &= k\sum_{i=1}^{n-1} [E(X_{i+1}^2) + E(X_i^2) - 2E(X_i X_{i+1})] \\ &= k\sum_{i=1}^{n-1} (\sigma^2 + \mu^2 + \sigma^2 + \mu^2 - 2\mu^2) \\ &= k(n-1)2\sigma^2 \end{aligned}$$
Therefore, $k = \frac{1}{2(n-1)}$

7-71 a.)The traditional estimate of the standard deviation, *S*, is 3.26. The mean of the sample is 13.43 so the values of $\left|X_i - \overline{X}\right|$ corresponding to the given observations are 3.43, 1.43, 4.43, 0.57, 4.57, 1.57 and 2.57. The median of these new quantities is 2.57 so the new estimate of the standard deviation is 3.81; slightly larger than the value obtained with the traditional estimator.

b.) Making the first observation in the original sample equal to 50 produces the following results. The traditional estimator, *S*, is equal to 13.91. The new estimator remains unchanged.

7-72 a.)

$$\begin{aligned} T_r = X_1 &+ \\ &X_1 + X_2 - X_1 + \\ &X_1 + X_2 - X_1 + X_3 - X_2 + \\ &\ldots + \\ &X_1 + X_2 - X_1 + X_3 - X_2 + \ldots + X_r - X_{r-1} + \\ &(n-r)(X_1 + X_2 - X_1 + X_3 - X_2 + \ldots + X_r - X_{r-1}) \end{aligned}$$

Because X_1 is the minimum lifetime of *n* items, $E(X_1) = \frac{1}{n\lambda}$.

Then, $X_2 - X_1$ is the minimum lifetime of (*n*-1) items from the memoryless property of the exponential and $E(X_2 - X_1) = \frac{1}{(n-1)\lambda}$.

Similarly, $E(X_k - X_{k-1}) = \frac{1}{(n-k+1)\lambda}$. Then,

$E(T_r) = \frac{n}{n\lambda} + \frac{n-1}{(n-1)\lambda} + \ldots + \frac{n-r+1}{(n-r+1)\lambda} = \frac{r}{\lambda}$ and $E\left(\frac{T_r}{r}\right) = \frac{1}{\lambda} = \mu$

b.) $V(T_r / r) = 1/(\lambda^2 r)$ is related to the variance of the Erlang distribution $V(X) = r/\lambda^2$. They are related by the value $(1/r^2)$. The censored variance is $(1/r^2)$ times the uncensored variance.

Section 7.3.3 on CD

S7-1 From Example S7-2 the posterior distribution for μ is normal with mean $\dfrac{(\sigma^2/n)\mu_0+\sigma_0^2\bar{x}}{\sigma_0^2+\sigma^2/n}$ and variance $\dfrac{\sigma_0^2/(\sigma^2/n)}{\sigma_0^2+\sigma^2/n}$.

The Bayes estimator for μ goes to the MLE as n increases. This follows since σ^2/n goes to 0, and the estimator approaches $\dfrac{\sigma_0^2\bar{x}}{\sigma_0^2}$ (the σ_0^2's cancel). Thus, in the limit $\hat{\mu}=\bar{x}$.

S7-2 a) Because $f(x\mid\mu)=\dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ and $f(\mu)=\dfrac{1}{b-a}$ for $a\le\mu\le b$, the joint distribution is $f(x,\mu)=\dfrac{1}{(b-a)\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $-\infty<x<\infty$ and $a\le\mu\le b$. Then,

$f(x)=\dfrac{1}{b-a}\displaystyle\int_a^b\dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}d\mu$ and this integral is recognized as a normal probability. Therefore,

$f(x)=\dfrac{1}{b-a}\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]$ where Φ(x) is the standard normal cumulative distribution function.

Then, $f(\mu\mid x)=\dfrac{f(x,\mu)}{f(x)}=\dfrac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]}$

b) The Bayes estimator is $\tilde{\mu}=\displaystyle\int_a^b\dfrac{\mu e^{-\frac{(x-\mu)^2}{2\sigma^2}}d\mu}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]}$.

Let v = (x - μ). Then, dv = - dμ and

$$\tilde{\mu}=\int_{x-b}^{x-a}\frac{(x-v)e^{-\frac{v^2}{2\sigma^2}}dv}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]}$$

$$=\frac{x\left[\Phi\left(\frac{x-a}{\sigma}\right)-\Phi\left(\frac{x-b}{\sigma}\right)\right]}{\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]}-\int_{x-b}^{x-a}\frac{ve^{-\frac{v^2}{2\sigma^2}}dv}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]}$$

Let w = $\dfrac{v^2}{2\sigma^2}$. Then, $dw=[\frac{2v}{2\sigma^2}]dv=[\frac{v}{\sigma^2}]dv$ and

$$\tilde{\mu}=x-\int_{\frac{(x-b)^2}{2\sigma^2}}^{\frac{(x-a)^2}{2\sigma^2}}\frac{\sigma e^{-w}dw}{\sqrt{2\pi}\left[\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)\right]}$$

$$=x+\frac{\sigma}{\sqrt{2\pi}}\left[\frac{e^{-\frac{(x-a)^2}{2\sigma^2}}-e^{-\frac{(x-b)^2}{2\sigma^2}}}{\Phi\left(\frac{b-x}{\sigma}\right)-\Phi\left(\frac{a-x}{\sigma}\right)}\right]$$

S7-3. a) $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ for x = 0, 1, 2, and $f(\lambda) = \left(\dfrac{m+1}{\lambda_0}\right)^{m+1} \dfrac{\lambda^m e^{-(m+1)\frac{\lambda}{\lambda_0}}}{\Gamma(m+1)}$ for $\lambda > 0$. Then,

$$f(x,\lambda) = \frac{(m+1)^{m+1}\lambda^{m+x} e^{-\lambda-(m+1)\frac{\lambda}{\lambda_0}}}{\lambda_0^{m+1}\Gamma(m+1)x!}.$$

This last density is recognized to be a gamma density as a function of λ. Therefore, the posterior distribution of λ is a gamma distribution with parameters m + x + 1 and 1 + $\dfrac{m+1}{\lambda_0}$.

b) The mean of the posterior distribution can be obtained from the results for the gamma distribution to be

$$\frac{m+x+1}{\left[1+\frac{m+1}{\lambda_0}\right]} = \lambda_0\left(\frac{m+x+1}{m+\lambda_0+1}\right)$$

S7-4 a) From Example S7-2, the Bayes estimate is $\tilde{\mu} = \dfrac{\frac{9}{25}(4)+1(4.85)}{\frac{9}{25}+1} = 4.625$

b.) $\hat{\mu} = \bar{x} = 4.85$ The Bayes estimate appears to underestimate the mean.

S7-5. a) From Example S7-2, $\tilde{\mu} = \dfrac{(0.01)(5.03)+(\frac{1}{25})(5.05)}{0.01+\frac{1}{25}} = 5.046$

b.) $\hat{\mu} = \bar{x} = 5.05$ The Bayes estimate is very close to the MLE of the mean.

S7-6. a) $f(x \mid \lambda) = \lambda e^{-\lambda x},\ x \ge 0$ and $f(\lambda) = 0.01e^{-0.01\lambda}$. Then,

$f(x_1, x_2, \lambda) = \lambda^2 e^{-\lambda(x_1+x_2)}0.01e^{-0.01\lambda} = 0.01\lambda^2 e^{-\lambda(x_1+x_2+0.01)}$. As a function of λ, this is recognized as a gamma density with parameters 3 and $x_1 + x_2 + 0.01$. Therefore, the posterior mean for λ is

$$\tilde{\lambda} = \frac{3}{x_1+x_2+0.01} = \frac{3}{2\bar{x}+0.01} = 0.00133.$$

b) Using the Bayes estimate for λ, P(X<1000)= $\int_0^{1000} 0.00133e^{-.00133x}dx$ =0.736.