CHAPTER 5

Section 5-1

5-1. First, f(x,y) ≥ 0. Let R denote the range of (X,Y).
Then, $\sum_R f(x,y) = \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = 1$

5-2 a) P(X < 2.5, Y < 3) = f(1.5,2)+f(1,1) = 1/8+1/4=3/8
b) P(X < 2.5) = f (1.5, 2) + f (1.5, 3) +f(1,1)= 1/8 + 1/4+1/4 = 5/8
c) P(Y < 3) = f (1.5, 2)+f(1,1) = 1/8+1/4=3/8
d) P(X > 1.8, Y > 4.7) = f (3, 5) = 1/8

5-3. E(X) = 1(1/4)+ 1.5(3/8) + 2.5(1/4) + 3(1/8) = 1.8125
E(Y) = 1(1/4)+2(1/8) + 3(1/4) + 4(1/4) + 5(1/8) = 2.875

5-4 a) marginal distribution of *X*

x	f(x)
1	1/4
1.5	3/8
2.5	1/4
3	1/8

b) $f_{Y|1.5}(y) = \dfrac{f_{XY}(1.5, y)}{f_X(1.5)}$ and $f_X(1.5)$ = 3/8. Then,

y	$f_{Y\|1.5}(y)$
2	(1/8)/(3/8)=1/3
3	(1/4)/(3/8)=2/3

c) $f_{X|2}(x) = \dfrac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2)$ = 1/8. Then,

x	$f_{X\|2}(y)$
1.5	(1/8)/(1/8)=1

d) $E(Y|X=1.5)$ = 2(1/3)+3(2/3) =2 1/3
e) Since $f_{Y|1.5}(y) \neq f_Y(y)$, X and Y are not independent

5-5 Let R denote the range of (X,Y). Because
$\sum_R f(x,y) = c(2+3+4+3+4+5+4+5+6) = 1$, 36c = 1, and c = 1/36

5-6. a) $P(X = 1, Y < 4) = f_{XY}(1,1) + f_{XY}(1,2) + f_{XY}(1,3) = \frac{1}{36}(2+3+4) = 1/4$
b) P(X = 1) is the same as part a. = 1/4
c) $P(Y = 2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{1}{36}(3+4+5) = 1/3$
d) $P(X < 2, Y < 2) = f_{XY}(1,1) = \frac{1}{36}(2) = 1/18$

5-7.

$$E(X) = 1[f_{XY}(1,1) + f_{XY}(1,2) + f_{XY}(1,3)] + 2[f_{XY}(2,1) + f_{XY}(2,2) + f_{XY}(2,3)]$$
$$+ \ 3[f_{XY}(3,1) + f_{XY}(3,2) + f_{XY}(3,3)]$$
$$= \left(1\times\tfrac{9}{36}\right) + \left(2\times\tfrac{12}{36}\right) + \left(3\times\tfrac{15}{36}\right) = 13/6 = 2.167$$
$$V(X) = (1-\tfrac{13}{6})^2\ \tfrac{9}{36} + (2-\tfrac{13}{6})^2\ \tfrac{12}{36} + (3-\tfrac{13}{6})^2\ \tfrac{15}{36} = 0.639$$
$$E(Y) = 2.167$$
$$V(Y) = 0.639$$

5-8 a) marginal distribution of *X*

x	$f_X(x) = f_{XY}(x,1) + f_{XY}(x,2) + f_{XY}(x,3)$
1	1/4
2	1/3
3	5/12

b) $f_{Y|X}(y) = \dfrac{f_{XY}(1,y)}{f_X(1)}$

y	$f_{Y\|X}(y)$
1	(2/36)/(1/4)=2/9
2	(3/36)/(1/4)=1/3
3	(4/36)/(1/4)=4/9

c) $f_{X|Y}(x) = \dfrac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \tfrac{12}{36} = 1/3$

x	$f_{X\|Y}(x)$
1	(3/36)/(1/3)=1/4
2	(4/36)/(1/3)=1/3
3	(5/36)/(1/3)=5/12

d) $E(Y|X=1) = 1(2/9)+2(1/3)+3(4/9) = 20/9$

e) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

5-9. $f(x,y) \geq 0$ *and* $\sum_R f(x,y) = 1$

5-10. a) $P(X < 0.5, Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \tfrac{1}{8} + \tfrac{1}{4} = \tfrac{3}{8}$

b) $P(X < 0.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \tfrac{3}{8}$

c) $P(Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) + f_{XY}(0.5,1) = \tfrac{7}{8}$

d) $P(X > 0.25, Y < 4.5) = f_{XY}(0.5,1) + f_{XY}(1,2) = \tfrac{5}{8}$

5-11.

$$E(X) = -1(\tfrac{1}{8}) - 0.5(\tfrac{1}{4}) + 0.5(\tfrac{1}{2}) + 1(\tfrac{1}{8}) = \tfrac{1}{8}$$
$$E(Y) = -2(\tfrac{1}{8}) - 1(\tfrac{1}{4}) + 1(\tfrac{1}{2}) + 2(\tfrac{1}{8}) = \tfrac{1}{4}$$

5-12 a) marginal distribution of *X*

x	$f_X(x)$
-1	1/8
-0.5	¼
0.5	½
1	1/8

b) $f_{Y|X}(y) = \dfrac{f_{XY}(1,y)}{f_X(1)}$

y	$f_{Y\|X}(y)$
2	1/8/(1/8)=1

c) $f_{X|Y}(x) = \dfrac{f_{XY}(x,1)}{f_Y(1)}$

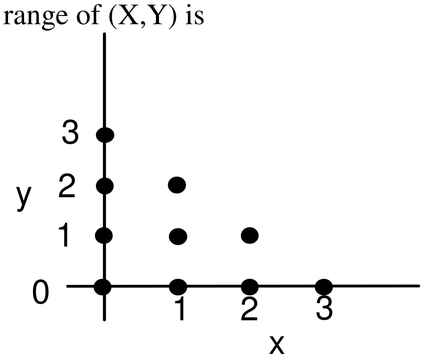
x	$f_{X\|Y}(x)$
0.5	½/(1/2)=1

d) $E(Y|X=1) = 0.5$

e) no, *X* and *Y* are not independent

5-13. Because X and Y denote the number of printers in each category,
$X \geq 0,\ Y \geq 0\ and\ X + Y = 4$

5-14. a) The range of (X,Y) is



Let H = 3, M = 2, and L = 1 denote the events that a bit has high, moderate, and low distortion, respectively. Then,

x,y	$f_{xy}(x,y)$
0,0	0.85738
0,1	0.1083
0,2	0.00456
0,3	0.000064
1,0	0.027075
1,1	0.00228
1,2	0.000048
2,0	0.000285
2,1	0.000012
3,0	0.000001

b)

x	$f_x(x)$
0	0.970299
1	0.029835
2	0.000297
3	0.000001

c) $f_{Y|1}(y) = \dfrac{f_{XY}(1, y)}{f_X(1)}$, $f_x(1) = 0.29835$

y	$f_{Y|1}(x)$
0	0.09075
1	0.00764
2	0.000161

E(X)=0(0.970299)+1(0.029403)+2(0.000297)+3*(0.000001)=0.03 (or np=3*.01).

d) $f_{Y|1}(y) = \dfrac{f_{XY}(1, y)}{f_X(1)}$, $f_x(1) = 0.029403$

y	$f_{Y|1}(x)$
0	0.920824
1	0.077543
2	0.001632

e) E(Y|X=1)=0(.920824)+1(0.077543)+2(0.001632)=0.080807

f) No, X and Y are not independent since, for example, $f_Y(0) \neq f_{Y|1}(0)$.

5-15 a) The range of (X,Y) is $X \geq 0,\ Y \geq 0\ and\ X + Y \leq 4$. X is the number of pages with moderate graphic content and Y is the number of pages with high graphic output out of 4.

	x=0	x=1	x=2	x=3	x=4
y=4	$5.35\text{x}10^{-05}$	0	0	0	0
y=3	0.00184	0.00092	0	0	0
y=2	0.02031	0.02066	0.00499	0	0
y=1	0.08727	0.13542	0.06656	0.01035	0
y=0	0.12436	0.26181	0.19635	0.06212	0.00699

b.)

	x=0	x=1	x=2	x=3	x=4
f(x)	0.2338	0.4188	0.2679	0.0725	0.0070

c.)
E(X)=

$$\sum_{0}^{4} x_i f(x_i) = 0(0.2338) + 1(0.4188) + 2(0.2679) + 3(0.7248) = 4(0.0070) = 1.2$$

d.) $f_{Y|3}(y) = \dfrac{f_{XY}(3, y)}{f_X(3)}$, $f_x(3) = 0.0725$

y	$f_{Y\|3}(y)$
0	0.857
1	0.143
2	0
3	0
4	0

e) E(Y|X=3) = 0(0.857)+1(0.143) = 0.143
f) V(Y|X=3) = $0^2(0.857)+1^2(0.143)- 0.143^2= 0.123$
g) no, *X* and *Y* are not independent

5-16 a) The range of (X,Y) is $X \geq 0,\ Y \geq 0\ and\ X + Y \leq 4$. X is the number of defective items found with inspection device 1 and Y is the number of defective items found with inspection device 2.

	x=0	x=1	x=2	x=3	x=4
y=0	$1.94\text{x}10^{-19}$	$1.10\text{x}10^{-16}$	$2.35\text{x}10^{-14}$	$2.22\text{x}10^{-12}$	$7.88\text{x}10^{-11}$
y=1	$2.59\text{x}10^{-16}$	$1.47\text{x}10^{-13}$	$3.12\text{x}10^{-11}$	$2.95\text{x}10^{-9}$	$1.05\text{x}10^{-7}$
y=2	$1.29\text{x}10^{-13}$	$7.31\text{x}10^{-11}$	$1.56\text{x}10^{-8}$	$1.47\text{x}10^{-6}$	$5.22\text{x}10^{-5}$
y=3	$2.86\text{x}10^{-11}$	$1.62\text{x}10^{-8}$	$3.45\text{x}10^{-6}$	$3.26\text{x}10^{-4}$	0.0116
y=4	$2.37\text{x}10^{-9}$	$1.35\text{x}10^{-6}$	$2.86\text{x}10^{-4}$	0.0271	0.961

$$f(x,y) = \left[\binom{4}{x}(0.993)^{x}(0.007)^{4-x}\right]\left[\binom{4}{y}(0.997)^{y}(0.003)^{4-y}\right]$$

For x=1,2,3,4 and y=1,2,3,4

b.)

$$f(x,y) = \left[\binom{4}{x}(0.993)^{x}(0.007)^{4-x}\right] \text{ for } x = 1,2,3,4$$

	x=0	x=1	x=2	x=3	x=4
f(x)	2.40×10^{-9}	1.36×10^{-6}	2.899×10^{-4}	0.0274	0.972

c.)since x is binomial $E(X) = n(p) = 4*(0.993) = 3.972$

d.) $f_{Y|2}(y) = \dfrac{f_{XY}(2,y)}{f_X(2)} = f(y)$, $f_x(2) = 0.0725$

y	$f_{Y|1}(y)$=f(y)
0	8.1×10^{-11}
1	1.08×10^{-7}
2	5.37×10^{-5}
3	0.0119
4	0.988

e) E(Y|X=2) = E(Y)= $n(p)$= 4(0.997)=3.988

f) V(Y|X=2) = V(Y)=$n(p)(1-p)$=4(0.997)(0.003)=0.0120

g) yes, X and Y are independent.

Section 5-2

5-17. a) $P(X=2) = f_{XYZ}(2,1,1) + f_{XYZ}(2,1,2) + f_{XYZ}(2,2,1) + f_{XYZ}(2,2,2) = 0.5$

b) $P(X=1, Y=2) = f_{XYZ}(1,2,1) + f_{XYZ}(1,2,2) = 0.35$

c) c) $P(Z<1.5) = f_{XYZ}(1,1,1) + f_{XYZ}(1,2,1) + f_{XYZ}(2,1,1) + f_{XYZ}(2,2,1) = 0.5$

d)

$P(X=1 \; or \; Z=2) = P(X=1) + P(Z=2) - P(X=1, Z=2) = 0.5 + 0.5 - 0.3 = 0.7$

e) E(X) = 1(0.5) + 2(0.5) = 1.5

5-18 a) $P(X=1 \mid Y=1) = \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{0.05 + 0.10}{0.15 + 0.2 + 0.1 + 0.05} = 0.3$

b) $P(X=1, Y=1 \mid Z=2) == \dfrac{P(X=1, Y=1, Z=2)}{P(Z=2)} = \dfrac{0.1}{0.1 + 0.2 + 0.15 + 0.05} = 0.2$

c) $P(X=1 \mid Y=1, Z=2) = \dfrac{P(X=1, Y=1, Z=2)}{P(Y=1, Z=2)} = \dfrac{0.10}{0.10 + 0.15} = 0.4$

5-19. $f_{X|YZ}(x) = \dfrac{f_{XYZ}(x,1,2)}{f_{YZ}(1,2)}$ and $f_{YZ}(1,2) = f_{XYZ}(1,1,2) + f_{XYZ}(2,1,2) = 0.25$

x	$f_{X\|YZ}(x)$
1	0.10/0.25=0.4
2	0.15/0.25=0.6

5-20 a.) percentage of slabs classified as high = p_1 = 0.05
percentage of slabs classified as medium = p_2 = 0.85
percentage of slabs classified as low = p_3 = 0.10

b.) X is the number of voids independently classified as high $X \geq 0$
Y is the number of voids independently classified as medium $Y \geq 0$
Z is the number of with a low number of voids and $Z \geq 0$
And $X+Y+Z = 20$

c.) p_1 is the percentage of slabs classified as high.

d) E(X)=np_1 = 20(0.05) = 1
V(X)=$np_1(1-p_1)$= 20(0.05)(0.95) = 0.95

5-21. a) $P(X=1,Y=17,Z=3)=0$ Because the point $(1,17,3)\neq 20$ is not in the range of (X,Y,Z).

b)
$$P(X\le 1,Y=17,Z=3)=P(X=0,Y=17,Z=3)+P(X=1,Y=17,Z=3)$$
$$=\frac{20!}{0!17!3!}0.05^{0}0.85^{17}0.10^{3}+0=0.07195$$

Because the point $(1,17,3)\neq 20$ is not in the range of (X,Y,Z).

c) Because X is binomial, $P(X\le 1)=\binom{20}{0}0.05^{0}0.95^{20}+\binom{20}{1}0.05^{1}0.95^{19}=0.7358$

d.) Because X is binomial E(X) = np = 20(0.05) = 1

5-22 a) The probability is 0 since x+y+z>20

$$P(X=2,Z=3\mid Y=17)=\frac{P(X=2,Z=3,Y=17)}{P(Y=17)}.$$

Because Y is binomial, $P(Y=17)=\binom{20}{17}0.85^{17}0.15^{3}=0.2428$.

Then, $P(X=2,Z=3,Y=17)=\dfrac{20!}{2!3!17!}\dfrac{0.05^{2}0.85^{17}0.10^{3}}{0.2054}=0$

b) $P(X=2\mid Y=17)=\dfrac{P(X=2,Y=17)}{P(Y=17)}$. Now, because x+y+z = 20,

P(X=2, Y=17) = P(X=2, Y=17, Z=1) = $\dfrac{20!}{2!17!1!}0.05^{2}0.85^{17}0.10^{1}=0.0540$

$$P(X=2\mid Y=17)=\frac{P(X=2,Y=17)}{P(Y=17)}=\frac{0.0540}{0.2428}=0.2224$$

c)
$$E(X\mid Y=17)=0\left(\frac{P(X=0,Y=17)}{P(Y=17)}\right)+1\left(\frac{P(X=1,Y=17)}{P(Y=17)}\right)$$
$$+2\left(\frac{P(X=2,Y=17)}{P(Y=17)}\right)+3\left(\frac{P(X=3,Y=17)}{P(Y=17)}\right)$$
$$E(X\mid Y=17)=0\left(\frac{0.07195}{0.2428}\right)+1\left(\frac{0.1079}{0.2428}\right)+2\left(\frac{0.05396}{0.2428}\right)+3\left(\frac{0.008994}{0.2428}\right)$$
$$=1$$

5-23. a) The range consists of nonnegative integers with $x+y+z=4$.
b) Because the samples are selected without replacement, the trials are not independent and the joint distribution is not multinomial.

5-24 $$P(X = x \mid Y = 2) = \frac{f_{XY}(x,2)}{f_Y(2)}$$

$$P(Y=2) = \frac{\binom{4}{0}\binom{5}{2}\binom{6}{2}}{\binom{15}{4}} + \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} + \frac{\binom{4}{2}\binom{5}{2}\binom{6}{0}}{\binom{15}{4}} = 0.1098 + 0.1758 + 0.0440 = 0.3296$$

$$P(X = 0 \text{ and } Y = 2) = \frac{\binom{4}{0}\binom{5}{2}\binom{6}{2}}{\binom{15}{4}} = 0.1098$$

$$P(X = 1 \text{ and } Y = 2) = \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} = 0.1758$$

$$P(X = 2 \text{ and } Y = 2) = \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} = 0.0440$$

x	$f_{XY}(x,2)$
0	0.1098/0.3296=0.3331
1	0.1758/0.3296=0.5334
2	0.0440/0.3296=0.1335

5-25. P(X=x, Y=y, Z=z) is the number of subsets of size 4 that contain x printers with graphics enhancements, y printers with extra memory, and z printers with both features divided by the number of subsets of size 4. From the results on the CD material on counting techniques, it can be shown that

$$P(X = x, Y = y, Z = z) = \frac{\binom{4}{x}\binom{5}{y}\binom{6}{z}}{\binom{15}{4}} \quad \text{for x+y+z = 4.}$$

a) $P(X = 1, Y = 2, Z = 1) = \frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}} = 0.1758$

b) $P(X = 1, Y = 1) = P(X = 1, Y = 1, Z = 2) = \frac{\binom{4}{1}\binom{5}{1}\binom{6}{2}}{\binom{15}{4}} = 0.2198$

c) The marginal distribution of X is hypergeometric with N = 15, n = 4, K = 4. Therefore, E(X) = nK/N = 16/15 and V(X) = 4(4/15)(11/15)[11/14] = 0.6146.

5-26 a)

$$P(X=1,Y=2 \mid Z=1) = P(X=1,Y=2,Z=1)/P(Z=1)$$
$$= \left[\frac{\binom{4}{1}\binom{5}{2}\binom{6}{1}}{\binom{15}{4}}\right] \Big/ \left[\frac{\binom{6}{1}\binom{9}{3}}{\binom{15}{4}}\right] = 0.4762$$

b)

$$P(X=2 \mid Y=2) = P(X=2,Y=2)/P(Y=2)$$
$$= \left[\frac{\binom{4}{2}\binom{5}{2}\binom{6}{0}}{\binom{15}{4}}\right] \Big/ \left[\frac{\binom{5}{2}\binom{10}{2}}{\binom{15}{4}}\right] = 0.1334$$

c) Because X+Y+Z = 4, if Y = 0 and Z = 3, then X = 1. Because X must equal 1, $f_{X|YZ}(1) = 1$.

5-27. a)The probability distribution is multinomial because the result of each trial (a dropped oven) results in either a major, minor or no defect with probability 0.6, 0.3 and 0.1 respectively. Also, the trials are independent

b.) Let X, Y, and Z denote the number of ovens in the sample of four with major, minor, and no defects, respectively.

$$P(X=2,Y=2,Z=0) = \frac{4!}{2!2!0!}0.6^2 0.3^2 0.1^0 = 0.1944$$

c.) $$P(X=0,Y=0,Z=4) = \frac{4!}{0!0!4!}0.6^0 0.3^0 0.1^4 = 0.0001$$

5-28 a.) $f_{XY}(x,y) = \sum_R f_{XYZ}(x,y,z)$ where R is the set of values for z such that x+y+z = 4. That is, R consists of the single value z = 4-x-y and

$$f_{XY}(x,y) = \frac{4!}{x!\,y!(4-x-y)!}0.6^x 0.3^y 0.1^{4-x-y} \quad for\ x+y \le 4.$$

b.) $E(X) = np_1 = 4(0.6) = 2.4$

c.) $E(Y) = np_2 = 4(0.3) = 1.2$

5-29 a.) $P(X=2 \mid Y=2) = \dfrac{P(X=2,Y=2)}{P(Y=2)} = \dfrac{0.1944}{0.2646} = 0.7347$

$P(Y=2) = \binom{4}{2}0.3^2 0.7^4 = 0.2646$ from the binomial marginal distribution of *Y*

b) Not possible, *x*+*y*+*z*=4, the probability is zero.

c.) $P(X \mid Y=2) = P(X=0 \mid Y=2), P(X=1 \mid Y=2), P(X=2 \mid Y=2)$

$$P(X=0 \mid Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \left(\frac{4!}{0!2!2!}0.6^0 0.3^2 0.1^2\right)\Big/ 0.2646 = 0.0204$$

$$P(X=1 \mid Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \left(\frac{4!}{1!2!1!}0.6^1 0.3^2 0.1^1\right)\Big/ 0.2646 = 0.2449$$

$$P(X=2 \mid Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \left(\frac{4!}{2!2!0!}0.6^2 0.3^2 0.1^0\right)\Big/ 0.2646 = 0.7347$$

d.) E(X|Y=2)=0(0.0204)+1(0.2449)+2(0.7347) = 1.7143

5-30 Let X,Y, and Z denote the number of bits with high, moderate, and low distortion. Then, the joint distribution of X, Y, and Z is multinomial with n =3 and $p_1 = 0.01,\ p_2 = 0.04,\ and\ p_3 = 0.95$.

a)

$$P(X=2, Y=1) = P(X=2, Y=1, Z=0)$$

$$= \frac{3!}{2!1!0!}0.01^2 0.04^1 0.95^0 = 1.2 \times 10^{-5}$$

b) $P(X=0, Y=0, Z=3) = \dfrac{3!}{0!0!3!}0.01^0 0.04^0 0.95^3 = 0.8574$

5-31 a., X has a binomial distribution with n = 3 and p = 0.01. Then, E(X) = 3(0.01) = 0.03 and V(X) = 3(0.01)(0.99) = 0.0297.

b. first find $P(X \mid Y=2)$

$$P(Y=2) = P(X=1, Y=2, Z=0) + P(X=0, Y=2, Z=1)$$

$$= \frac{3!}{1!2!0!}0.01(0.04)^2 0.95^0 + \frac{3!}{0!2!1!}0.01^0 (0.04)^2 0.95^1 = 0.0046$$

$$P(X=0 \mid Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \left(\frac{3!}{0!2!1!}0.01^0 0.04^2 0.95^1\right)\Big/ 0.004608 = 0.98958$$

$$P(X=1 \mid Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \left(\frac{3!}{1!2!1!}0.01^1 0.04^2 0.95^0\right)\Big/ 0.004608 = 0.01042$$

$$E(X \mid Y=2) = 0(0.98958) + 1(0.01042) = 0.01042$$

$$V(X \mid Y=2) = E(X^2) - (E(X))^2 = 0.01042 - (0.01042)^2 = 0.01031$$

5-32 a.) Let X,Y, and Z denote the risk of new competitors as no risk, moderate risk, and very high risk. Then, the joint distribution of X, Y, and Z is multinomial with n =12 and $p_1 = 0.13, \quad p_2 = 0.72, \quad and \quad p_3 = 0.15$. X, Y and Z ≥ 0 and x+y+z=12

b.) $P(X = 1, Y = 3, Z = 1) = 0$, not possible since x+y+z≠12

c)

$$P(Z \le 2) = \binom{12}{0} 0.15^0 0.85^{12} + \binom{12}{1} 0.15^1 0.85^{11} + \binom{12}{2} 0.15^2 0.85^{10}$$
$$= 0.1422 + 0.3012 + 0.2924 = 0.7358$$

5-33 a.) $P(Z = 2 \mid Y = 1, X = 10) = 0$

b.) first get

$$P(X = 10) = P(X = 10, Y = 2, Z = 0) + P(X = 10, Y = 1, Z = 1) + P(X = 10, Y = 0, Z = 2)$$
$$= \frac{12!}{10!2!0!} 0.13^{10} 0.72^2 0.15^0 + \frac{12!}{10!1!1!} 0.13^{10} 0.72^1 0.15^1 + \frac{12!}{10!0!2!} 0.13^{10} 0.72^0 0.15^2$$
$$= 4.72x10^{-8} + 1.97x10^{-8} + 2.04x10^{-9} = 6.89x10^{-8}$$
$$P(Z \le 1 \mid X = 10) = \frac{P(Z = 0, Y = 2, X = 10)}{P(X = 10)} + \frac{P(Z = 1, Y = 1, X = 10)}{P(X = 10)}$$
$$= \frac{12!}{10!2!0!} 0.13^{10} 0.72^2 0.15^0 \Big/ 6.89x10^{-8} + \frac{12!}{10!1!1!} 0.13^{10} 0.72^1 0.15^1 \Big/ 6.89x10^{-8}$$
$$= 0.9698$$

c.)

$$P(Y \le 1, Z \le 1 \mid X = 10) = \frac{P(Z = 1, Y = 1, X = 10)}{P(X = 10)}$$
$$= \frac{12!}{10!1!1!} 0.13^{10} 0.72^1 0.15^1 \Big/ 6.89x10^{-8}$$
$$= 0.2852$$

d.

$$E(Z \mid X = 10) = (0(4.72x10^{-8}) + 1(1.97x10^{-8}) + 2(2.04x10^{-9})) / 6.89x10^{-8}$$
$$= 2.378x10^{-8}$$

Section 5-3

5-34 Determine c such that $c\int_0^3\int_0^3 xydxdy = c\int_0^3 y\frac{x^2}{2}\Big|_0^3 dy = c(4.5\frac{y^2}{2}\Big|_0^3) = \frac{81}{4}c.$

Therefore, $c = 4/81$.

5-35. a) $P(X<2, Y<3) = \frac{4}{81}\int_0^3\int_0^2 xydxdy = \frac{4}{81}(2)\int_0^3 ydy = \frac{4}{81}(2)(\frac{9}{2}) = 0.4444$

b) P(X < 2.5) = P(X < 2.5, Y < 3) because the range of Y is from 0 to 3.

$$P(X<2.5, Y<3) = \frac{4}{81}\int_0^3\int_0^{2.5} xydxdy = \frac{4}{81}(3.125)\int_0^3 ydy = \frac{4}{81}(3.125)\frac{9}{2} = 0.6944$$

c) $P(1<Y<2.5) = \frac{4}{81}\int_1^{2.5}\int_0^3 xydxdy = \frac{4}{81}(4.5)\int_1^{2.5} ydy = \frac{18}{81}\frac{y^2}{2}\Big|_1^{2.5}$ =0.5833

5-35 d)

$$P(X>1.8, 1<Y<2.5) = \frac{4}{81}\int_1^{2.5}\int_{1.8}^3 xydxdy = \frac{4}{81}(2.88)\int_1^{2.5} ydy = \frac{4}{81}(2.88)\frac{(2.5^2-1)}{2} = 0.3733$$

e) $E(X) = \frac{4}{81}\int_0^3\int_0^3 x^2ydxdy = \frac{4}{81}\int_0^3 9ydy = \frac{4}{9}\frac{y^2}{2}\Big|_0^3 = 2$

f) $P(X<0, Y<4) = \frac{4}{81}\int_0^4\int_0^0 xydxdy = 0\int_0^4 ydy = 0$

5-36 a) $f_X(x) = \int_0^3 f_{XY}(x,y)dy = x\frac{4}{81}\int_0^3 ydy = \frac{4}{81}x(4.5) = \frac{2x}{9}$ $\quad for \quad 0<x<3$.

b) $f_{Y|1.5}(y) = \dfrac{f_{XY}(1.5, y)}{f_X(1.5)} = \dfrac{\frac{4}{81}y(1.5)}{\frac{2}{9}(1.5)} = \frac{2}{9}y$ for $0 < y < 3$.

c) E(Y|X=1.5) = $\int_0^3 y\left(\frac{2}{9}y\right)dy = \frac{2}{9}\int_0^3 y^2dy = \frac{2y^3}{27}\Big|_0^3 = 6$

d.) $P(Y<2 \mid X=1.5) = f_{Y|1.5}(y) = \int_0^2 \frac{2}{9}ydy = \frac{1}{9}y^2\Big|_0^2 = \frac{4}{9} - 0 = \frac{4}{9}$

e) $f_{X|2}(x) = \dfrac{f_{XY}(x,2)}{f_Y(2)} = \dfrac{\frac{4}{81}x(2)}{\frac{2}{9}(2)} = \frac{2}{9}x$ for 0 < x < 3.

5-37.

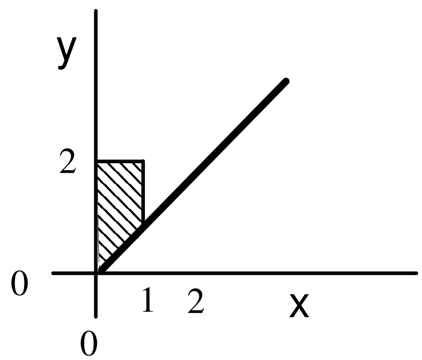
$$c\int_0^3 \int_x^{x+2} (x+y)\,dydx = \int_0^3 xy + \tfrac{y^2}{2}\Big|_x^{x+2} dx$$

$$= \int_0^3 \left[x(x+2) + \tfrac{(x+2)^2}{2} - x^2 - \tfrac{x^2}{2}\right] dx$$

$$= c\int_0^3 (4x+2)\,dx = \left[2x^2 + 2x\right]_0^3 = 24c$$

Therefore, c = 1/24.

5-38 a) P(X < 1, Y < 2) equals the integral of $f_{XY}(x, y)$ over the following region.

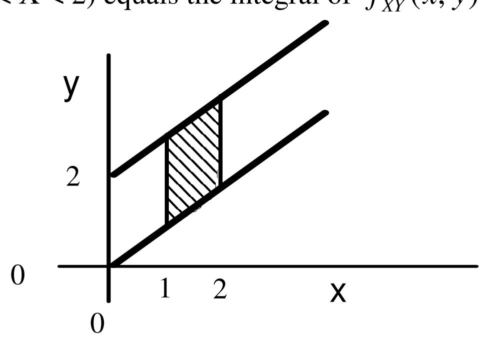


Then,

$$P(X < 1, Y < 2) = \frac{1}{24}\int_0^1 \int_x^2 (x+y)\,dydx = \frac{1}{24}\int_0^1 xy + \tfrac{y^2}{2}\Big|_x^2 dx = \frac{1}{24}\int_0^3 2x + 2 - \tfrac{3x^2}{2} dx =$$

$$\frac{1}{24}\left[x^2 + 2x - \tfrac{x^3}{2}\Big|_0^1\right] = 0.10417$$

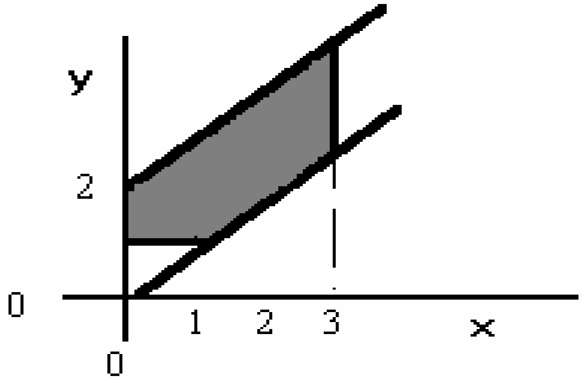
b) P(1 < X < 2) equals the integral of $f_{XY}(x, y)$ over the following region.



$$P(1 < X < 2) = \frac{1}{24}\int_1^2 \int_x^{x+2} (x+y)\,dydx = \frac{1}{24}\int_1^2 xy + \tfrac{y^2}{2}\Big|_x^{x+2} dx$$

$$= \frac{1}{24}\int_0^3 (4x+2)\,dx = \frac{1}{24}\left[2x^2 + 2x\Big|_1^2\right] = \tfrac{1}{6}.$$

c) P(Y > 1) is the integral of $f_{XY}(x, y)$ over the following region.

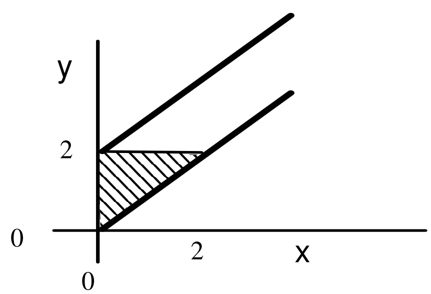


$$P(Y>1)=1-P(Y\le 1)=1-\tfrac{1}{24}\int_0^1\int_x^1 (x+y)dydx=1-\frac{1}{24}\int_0^1 (xy+\frac{y^2}{2})\Big|_x^1$$

$$=1-\frac{1}{24}\int_0^1 x+\frac{1}{2}-\frac{3}{2}x^2dx=1-\frac{1}{24}\left(\frac{x^2}{2}+\frac{1}{2}-\frac{1}{2}x^3\right)\Bigg|_0^1$$

$$=1-0.02083=0.9792$$

d) P(X < 2, Y < 2) is the integral of $f_{XY}(x,y)$ over the following region.



$$E(X)=\frac{1}{24}\int_0^3\int_x^{x+2} x(x+y)dydx=\frac{1}{24}\int_0^3 x^2y+\tfrac{xy^2}{2}\Big|_x^{x+2}dx$$

$$=\frac{1}{24}\int_0^3 (4x^2+2x)dx=\frac{1}{24}\left[\frac{4x^3}{3}+x^2\Big|_0^3\right]=\frac{15}{8}$$

e)

$$E(X)=\frac{1}{24}\int_0^3\int_x^{x+2} x(x+y)dydx=\frac{1}{24}\int_0^3 x^2y+\tfrac{xy^2}{2}\Big|_x^{x+2}dx$$

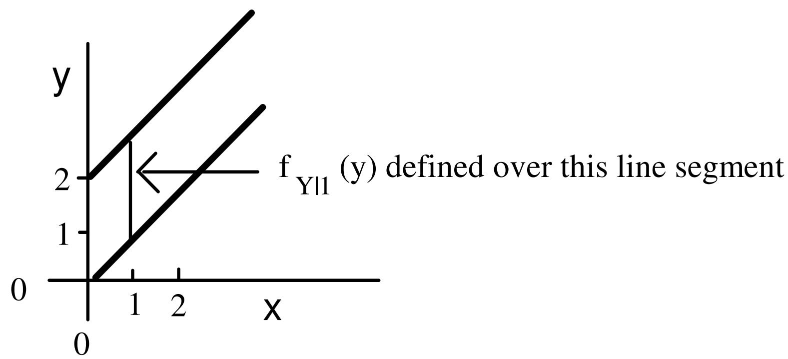
$$=\frac{1}{24}\int_0^3 (3x^2+2x)dx=\frac{1}{24}\left[x^3+x^2\Big|_0^3\right]=\frac{15}{8}$$

5-39. a) $f_X(x)$ is the integral of $f_{XY}(x,y)$ over the interval from x to x+2. That is,

$$f_X(x) = \frac{1}{24}\int_x^{x+2}(x+y)dy = \frac{1}{24}\left[xy + \frac{y^2}{2}\Big|_x^{x+2}\right] = \frac{x}{6} + \frac{1}{12} \quad \text{for } 0 < x < 3.$$

b) $f_{Y|1}(y) = \frac{f_{XY}(1,y)}{f_X(1)} = \frac{\frac{1}{24}(1+y)}{\frac{1}{6}+\frac{1}{12}} = \frac{1+y}{6}$ for 1 < y < 3.

See the following graph,

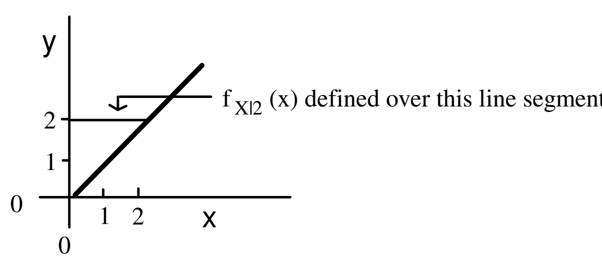


c) $E(Y|X=1) = \int_1^3 y\left(\frac{1+y}{6}\right)dy = \frac{1}{6}\int_1^3 (y+y^2)dy = \frac{1}{6}\left(\frac{y^2}{2}+\frac{y^3}{3}\right)\Big|_1^3$ =2.111

d.) $P(Y > 2 \mid X = 1) = \int_2^3 \left(\frac{1+y}{6}\right)dy = \frac{1}{6}\int_2^3 (1+y)dy = \frac{1}{6}\left(y+\frac{y^2}{2}\right)\Big|_2^3$ =0.5833

e.) $f_{X|2}(x) = \frac{f_{XY}(x,2)}{f_Y(2)}$. Here $f_Y(y)$ is determined by integrating over *x*. There are three regions of integration. For $0 < y \le 2$ the integration is from 0 to *y*. For $2 < y \le 3$ the integration is from *y*-2 to *y*. For $3 < y < 5$ the integration is from *y* to 3. Because the condition is *y*=2, only the first integration is needed.

$$f_Y(y) = \frac{1}{24}\int_0^y (x+y)dx = \frac{1}{24}\left[\frac{x^2}{2} + xy\Big|_0^y\right] = \frac{y^2}{16} \quad \text{for } 0 < y \le 2.$$



Therefore, $f_Y(2) = 1/4$ and $f_{X|2}(x) = \frac{\frac{1}{24}(x+2)}{1/4} = \frac{x+2}{6}$ for 0 < x < 2

5-40 $c\int_0^3\int_0^x xydydx = c\int_0^3 x\frac{y^2}{2}\bigg|_0^x dx = c\int_0^3 \frac{x^3}{2}dx\frac{x^4}{8} = \frac{81}{8}c$. Therefore, c = 8/81

5-41. a) P(X<1,Y<2)= $\frac{8}{81}\int_0^1\int_0^x xydydx = \frac{8}{81}\int_0^1 \frac{x^3}{2}dx = \frac{8}{81}\left(\frac{1}{8}\right) = \frac{1}{81}$.

b) P(1<X<2) = $\frac{8}{81}\int_1^2\int_0^x xydydx = \frac{8}{81}\int_1^2 x\frac{x^2}{2}dx = \left(\frac{8}{81}\right)\frac{x^4}{8}\bigg|_1^2 = \left(\frac{8}{81}\right)\frac{(2^4-1)}{8} = \frac{5}{27}$.

c)

$$P(Y>1) = \frac{8}{81}\int_1^3\int_1^x xydydx = \frac{8}{81}\int_1^3 x\left(\frac{x^2-1}{2}\right)dx = \frac{8}{81}\int_1^3 \frac{x^3}{2} - \frac{x}{2}dx = \frac{8}{81}\left(\frac{x^4}{8} - \frac{x^2}{4}\right)\bigg|_1^3$$

$$= \frac{8}{81}\left[\left(\frac{3^4}{8} - \frac{3^2}{4}\right) - \left(\frac{1^4}{8} - \frac{1^2}{4}\right)\right] = \frac{1}{81} = 0.01235$$

d) P(X<2,Y<2) = $\frac{8}{81}\int_0^2\int_0^x xydydx = \frac{8}{81}\int_0^2 \frac{x^3}{2}dx = \frac{8}{81}\left(\frac{2^4}{8}\right) = \frac{16}{81}$.

e.)

$$E(X) = \frac{8}{81}\int_0^3\int_0^x x(xy)dydx = \frac{8}{81}\int_0^3\int_0^x x^2ydydx = \frac{8}{81}\int_0^3 \frac{x^2}{2}x^2dx = \frac{8}{81}\int_0^3 \frac{x^4}{2}dx$$

$$= \left(\frac{8}{81}\right)\left(\frac{3^5}{10}\right) = \frac{12}{5}$$

f)

$$E(Y) = \frac{8}{81}\int_0^3\int_0^x y(xy)dydx = \frac{8}{81}\int_0^3\int_0^x xy^2dydx = \frac{8}{81}\int_0^3 x\frac{x^3}{3}dx$$

$$= \frac{8}{81}\int_0^3 \frac{x^4}{3}dx = \left(\frac{8}{81}\right)\left(\frac{3^5}{15}\right) = \frac{8}{5}$$

5-42 a.) $f(x) = \frac{8}{81}\int_0^x xy dy = \frac{4x^3}{81} \qquad 0 < x < 3$

b.) $f_{Y|x=1}(y) = \frac{f(1,y)}{f(1)} = \frac{\frac{8}{81}(1)y}{\frac{4(1)^3}{81}} = 2y \qquad 0 < y < 1$

c.) $E(Y \mid X = 1) = \int_0^1 2y dy = y^2\Big|_0^1 = 1$

d.) $P(Y > 2 \mid X = 1) = 0$ this isn't possible since the values of y are 0< $y < x$.

e.) $f(y) = \frac{8}{81}\int_0^3 xy dx = \frac{4y}{9}$, therefore

$$f_{X|Y=2}(x) = \frac{f(x,2)}{f(2)} = \frac{\frac{8}{81}x(2)}{\frac{4(2)}{9}} = \frac{2x}{9} \qquad 0 < x < 2$$

5-43. Solve for c

$$c\int_0^\infty \int_0^x e^{-2x-3y} dy dx = \frac{c}{3}\int_0^\infty e^{-2x}\left(1 - e^{-3x}\right)dx = \frac{c}{3}\int_0^\infty e^{-2x} - e^{-5x} dx =$$

$$\frac{c}{3}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{1}{10}c. \quad c = 10$$

5-44 a)

$$P(X < 1, Y < 2) = 10\int_0^1 \int_0^x e^{-2x-3y} dy dx = \frac{10}{3}\int_0^1 e^{-2x}(1 - e^{-3x})dy = \frac{10}{3}\int_0^1 e^{-2x} - e^{-5x} dy$$

$$= \frac{10}{3}\left(\frac{e^{-5x}}{5} - \frac{e^{-2x}}{2}\right)\Bigg|_0^1 = 0.77893$$

b)

$$P(1 < X < 2) = 10\int_1^2 \int_0^x e^{-2x-3y} dy dx = \frac{10}{3}\int_1^2 e^{-2x} - e^{-5x} dx$$

$$= \frac{10}{3}\left(\frac{e^{-5x}}{5} - \frac{e^{-2x}}{2}\right)\Bigg|_1^2 = 0.19057$$

c)

$$P(Y > 3) = 10\int_3^\infty \int_3^x e^{-2x-3y} dy dx = \frac{10}{3}\int_3^{1\infty} e^{-2x}(e^{-9} - e^{-3x})dy$$

$$= \frac{10}{3}\left(\frac{e^{-5x}}{5} - \frac{e^{-9}e^{-2x}}{2}\right)\Bigg|_3^\infty = 3.059x10^{-7}$$

d)

$$P(X<2,Y<2)=10\int_0^2\int_0^x e^{-2x-3y}dydx=\frac{10}{3}\int_0^2 e^{-2x}(1-e^{-3x})dx=\frac{10}{3}\left(\frac{e^{-10}}{5}-\frac{e^{-4}}{2}\right)\Bigg|_0^2$$

$$=0.9695$$

e) $E(X)=10\int_0^\infty\int_0^x xe^{-2x-3y}dydx=\frac{7}{10}$

f) $E(Y)=10\int_0^\infty\int_0^x ye^{-2x-3y}dydx=\frac{1}{5}$

5-45. a) $f(x)=10\int_0^x e^{-2x-3y}dy=\frac{10e^{-2z}}{3}(1-e^{-3x})=\frac{10}{3}(e^{-2x}-e^{-5x})$ for $0<x$

b) $f_{Y\backslash X=1}(y)=\frac{f_{X,Y}(1,y)}{f_X(1)}=\frac{10e^{-2-3y}}{\frac{10}{3}(e^{-2}-e^{-5})}=3.157e^{-3y}\quad 0<y<1$

c) $E(Y|X=1)=3.157\int_0^1 ye^{-3y}dy=0.2809$

d) $f_{X|Y=2}(x)\ =\ \frac{f_{X,Y}(x,2)}{f_Y(2)}=\frac{10e^{-2x-6}}{5e^{-10}}=2e^{-2x+4}$ for $2<x$,

where $f(y)=5e^{-5y}$ for $0<y$

5-46 $c\int_0^\infty\int_x^\infty e^{-2x}e^{-3y}dydx=\frac{c}{3}\int_0^\infty e^{-2x}(e^{-3x})dx=\frac{c}{3}\int_0^\infty e^{-5x}dx=\frac{1}{15}c$ $\qquad c=15$

5-47. a)

$$P(X<1,Y<2)=15\int_0^1\int_x^2 e^{-2x-3y}dydx=5\int_0^1 e^{-2x}(e^{-3x}-e^{-6})dx$$

$$=5\int_0^1 e^{-5x}dx-5e^{-6}\int_0^1 e^{-2x}dx=1-e^{-5}+\frac{5}{2}e^{-6}(e^{-2}-1)=0.9879$$

b) $P(1<X<2)=15\int_1^2\int_x^\infty e^{-2x-3y}dydx=5\int_1^2 e^{-5x}dydx=(e^{-5}-e^{10})=0.0067$

c)

$$P(Y>3)=15\left(\int_0^3\int_3^\infty e^{-2x-3y}dydx+\int_3^\infty\int_x^\infty e^{-2x-3y}dydx\right)=5\int_0^3 e^{-9}e^{-2x}dx+5\int_3^\infty e^{-5x}dx$$

$$=-\frac{3}{2}e^{-15}+\frac{5}{2}e^{-9}=0.000308$$

d)

$$P(X<2,Y<2)=15\int_0^2\int_x^2 e^{-2x-3y}dydx=5\int_0^2 e^{-2x}(e^{-3x}-e^{-6})dx=$$

$$=5\int_0^2 e^{-5x}dx-5e^{-6}\int_0^2 e^{-2x}dx=\left(1-e^{-10}\right)+\frac{5}{2}e^{-6}\left(e^{-4}-1\right)=0.9939$$

e) E(X) = $15\int_0^\infty\int_x^\infty xe^{-2x-3y}dydx=5\int_0^\infty xe^{-5x}dx=\frac{1}{5^2}=0.04$

f)

$$E(Y)=15\int_0^\infty\int_x^\infty ye^{-2x-3y}dydx=\frac{-3}{2}\int_0^\infty 5ye^{-5y}dy+\frac{5}{2}\int_0^\infty 3ye^{-3y}dy$$

$$=-\frac{3}{10}+\frac{5}{6}=\frac{8}{15}$$

5-48 a.) $f(x)=15\int_x^\infty e^{-2x-3y}dy=\frac{15}{3}(e^{-2z-3x})=5e^{-5x}$ for $x>0$

b) $f_X(1)=5e^{-5}$ $f_{XY}(1,y)=15e^{-2-3y}$

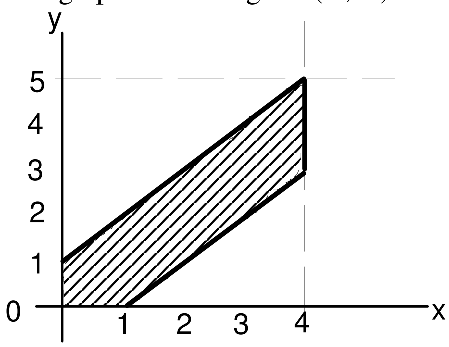
$$f_{Y|X=1}(y)=\frac{15e^{-2-3y}}{5e^{-5}}=3e^{3-3y}\text{ for }1<y$$

c) $E(Y\mid X=1)=\int_1^\infty 3ye^{3-3y}dy=-ye^{3-3y}\Big|_1^\infty+\int_1^\infty e^{3-3y}dy=4/3$

d) $\int_1^2 3e^{3-3y}dy=1-e^{-3}=0.9502$ for $0<y$, $f_Y(2)=\frac{15}{2}e^{-6}$

$$f_{X|Y=2}(y)=\frac{15e^{-2x-6}}{\frac{15}{2}e^{-6}}=2e^{-2x}$$

5-49. The graph of the range of (X, Y) is



$$\int_0^1 \int_0^{x+1} c\,dy\,dx + \int_1^4 \int_{x-1}^{x+1} c\,dy\,dx = 1$$

$$= c\int_0^1 (x+1)dx + 2c\int_1^4 dx$$

$$= \tfrac{3}{2}c + 6c = 7.5c = 1$$

Therefore, c = 1/7.5=2/15

5-50 a) $P(X < 0.5, Y < 0.5) = \int_0^{0.5}\int_0^{0.5} \frac{1}{7.5}\,dy\,dx = \frac{1}{30}$

b) $P(X < 0.5) = \int_0^{0.5}\int_0^{x+1} \frac{1}{7.5}\,dy\,dx = \frac{1}{7.5}\int_0^{0.5}(x+1)dx = \frac{2}{15}\left(\frac{5}{8}\right) = \frac{1}{12}$

c)

$$E(X) = \int_0^1\int_0^{x+1} \frac{x}{7.5}\,dy\,dx + \int_1^4\int_{x-1}^{x+1} \frac{x}{7.5}\,dy\,dx$$

$$= \frac{1}{7.5}\int_0^1 (x^2 + x)dx + \frac{2}{7.5}\int_1^4 (x)dx = \frac{12}{15}\left(\frac{5}{6}\right) + \frac{2}{7.5}(7.5) = \frac{19}{9}$$

d)

$$E(Y) = \frac{1}{7.5}\int_0^1\int_0^{x+1} y\,dy\,dx + \frac{1}{7.5}\int_1^4\int_{x-1}^{x+1} y\,dy\,dx$$

$$= \frac{1}{7.5}\int_0^1 \frac{(x+1)^2}{2}dx + \frac{1}{7.5}\int_1^4 \frac{(x+1)^2-(x-1)^2}{2}dx$$

$$= \frac{1}{15}\int_0^1 (x^2 + 2x + 1)dx + \frac{1}{15}\int_1^4 4x\,dx$$

$$= \frac{1}{15}\left(\frac{7}{3}\right) + \frac{1}{15}(30) = \frac{97}{45}$$

5-51. a.)

$$f(x) = \int_0^{x+1} \frac{1}{7.5} dy = \left(\frac{x+1}{7.5}\right) \quad \text{for} \quad 0 < x < 1,$$

$$f(x) = \int_{x-1}^{x+1} \frac{1}{7.5} dy = \left(\frac{x+1-(x-1)}{7.5}\right) = \frac{2}{7.5} \quad \text{for} \quad 1 < x < 4$$

b.)

$$f_{Y|X=1}(y) = \frac{f_{XY}(1, y)}{f_X(1)} = \frac{1/7.5}{2/7.5} = 0.5$$

$$f_{Y|X=1}(y) = 0.5 \quad \text{for } 0 < y < 2$$

c.) $E(Y \mid X = 1) = \int_0^2 \frac{y}{2} dy = \left.\frac{y^2}{4}\right|_0^2 = 1$

d.) $P(Y < 0.5 \mid X = 1) = \int_0^{0.5} 0.5 dy = 0.5y\Big|_0^{0.5} = 0.25$

5-52 Let X, Y, and Z denote the time until a problem on line 1, 2, and 3, respectively.

a)

$$P(X > 40, Y > 40, Z > 40) = [P(X > 40)]^3$$

because the random variables are independent with the same distribution. Now,

$P(X > 40) = \int_{40}^{\infty} \frac{1}{40} e^{-x/40} dx = -e^{-x/40}\Big|_{40}^{\infty} = e^{-1}$ and the answer is

$\left(e^{-1}\right)^3 = e^{-3} = 0.0498$.

b) $P(20 < X < 40, 20 < Y < 40, 20 < Z < 40) = [P(20 < X < 40)]^3$ and

$P(20 < X < 40) = -e^{-x/40}\Big|_{20}^{40} = e^{-0.5} - e^{-1} = 0.2387$.

The answer is $0.2387^3 = 0.0136$.

c.) The joint density is not needed because the process is represented by three independent exponential distributions. Therefore, the probabilities may be multiplied.

5-53 a.) μ=3.2 λ=1/3.2

$$P(X>5,Y>5)=(1/10.24)\int_5^\infty\int_5^\infty e^{-\frac{x}{3.2}-\frac{y}{3.2}}dydx = 3.2\int_5^\infty e^{-\frac{x}{3.2}}\left(e^{-\frac{5}{3.2}}\right)dx$$

$$=\left(e^{-\frac{5}{3.2}}\right)\left(e^{-\frac{5}{3.2}}\right)=0.0439$$

$$P(X>10,Y>10)=(1/10.24)\int_{10}^\infty\int_{10}^\infty e^{-\frac{x}{3.2}-\frac{y}{3.2}}dydx = 3.2\int_{10}^\infty e^{-\frac{x}{3.2}}\left(e^{-\frac{10}{3.2}}\right)dx$$

$$=\left(e^{-\frac{10}{3.2}}\right)\left(e^{-\frac{10}{3.2}}\right)=0.0019$$

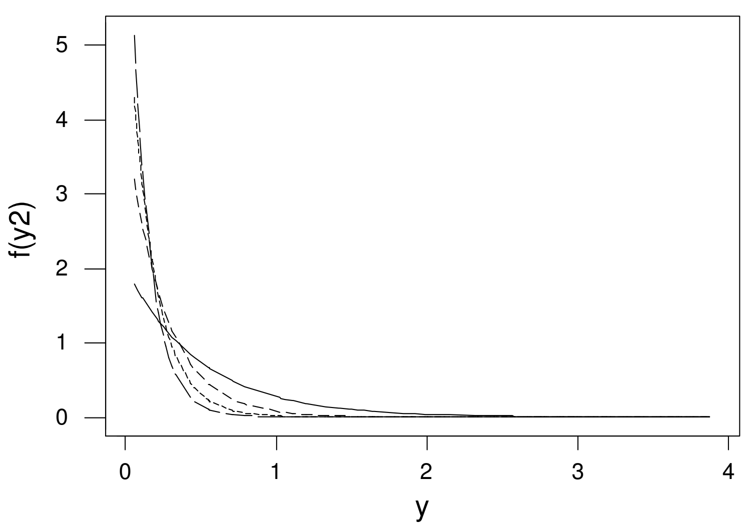
b.) Let *X* denote the number of orders in a 5-minute interval. Then *X* is a Poisson random variable with λ=5/3.2 = 1.5625.

$$P(X=2)=\frac{e^{-1.5625}(1.5625)^2}{2!}=0.256$$

For both systems, $P(X=2)P(Y=2)=0.256^2=0.0655$

c.) The joint probability distribution is not necessary because the two processes are independent and we can just multiply the probabilities.

5-54 a) $f_{Y|X=x}(y)$, for x = 2, 4, 6, 8



b) $P(Y<2 \mid X=2)=\int_0^2 2e^{-2y}dy=0.9817$

c) $E(Y \mid X=2)=\int_0^\infty 2ye^{-2y}dy=1/2$ (using integration by parts)

d) $E(Y \mid X=x)=\int_0^\infty xye^{-xy}dy=1/x$ (using integration by parts)

e) Use $f_X(x)=\dfrac{1}{b-a}=\dfrac{1}{10}$, $f_{Y|X}(x,y)=xe^{-xy}$, and the relationship $f_{Y|X}(x,y)=\dfrac{f_{XY}(x,y)}{f_X(x)}$

Therefore, $xe^{-xy}=\dfrac{f_{XY}(x,y)}{1/10}$ and $f_{XY}(x,y)=\dfrac{xe^{-xy}}{10}$

f) $f_Y(y)=\int_0^{10}\dfrac{xe^{-xy}}{10}dx=\dfrac{1-10ye^{-10y}-e^{-10y}}{10y^2}$ (using integration by parts)

Section 5-4

5-55. a) $P(X<0.5)=\int_0^{0.5}\int_0^1\int_0^1(8xyz)dzdydx=\int_0^{0.5}\int_0^1(4xy)dydx=\int_0^{0.5}(2x)dx=x^2\Big|_0^{0.5}=0.25$

b)

$$P(X<0.5,Y<0.5)=\int_0^{0.5}\int_0^{0.5}\int_0^1(8xyz)dzdydx$$

$$=\int_0^{0.5}\int_0^{0.5}(4xy)dydx=\int_0^{0.5}(0.5x)dx=\tfrac{x^2}{4}\Big|_0^{0.5}=0.0625$$

c) P(Z < 2) = 1, because the range of Z is from 0 to 1.

d) P(X < 0.5 or Z < 2) = P(X < 0.5) + P(Z < 2) - P(X < 0.5, Z < 2). Now, P(Z < 2) =1 and P(X < 0.5, Z < 2) = P(X < 0.5). Therefore, the answer is 1.

e) $E(X)=\int_0^1\int_0^1\int_0^1(8x^2yz)dzdydx=\int_0^1(2x^2)dx=\tfrac{2x^3}{3}\Big|_0^1=2/3$

5-56 a) $P(X<0.5 \mid Y=0.5)$ is the integral of the conditional density $f_{X|Y}(x)$. Now,

$f_{X|0.5}(x)=\dfrac{f_{XY}(x,0.5)}{f_Y(0.5)}$ and $f_{XY}(x,0.5)=\int_0^1(8xyz)dz=4xy$ for 0 < x < 1 and

0 < y < 1. Also, $f_Y(y)=\int_0^1\int_0^1(8xyz)dzdx=2y$ for 0 < y < 1.

Therefore, $f_{X|0.5}(x)=\dfrac{2x}{1}=2x$ for 0 < x < 1.

Then, $P(X<0.5 \mid Y=0.5)=\int_0^{0.5}2xdx=0.25$.

b) $P(X < 0.5, Y < 0.5 | Z = 0.8)$ is the integral of the conditional density of X and Y.

Now, $f_Z(z) = 2z$ for $0 < z < 1$ as in part a. and

$$f_{XY|Z}(x, y) = \frac{f_{XYZ}(x, y, z)}{f_Z(z)} = \frac{8xy(0.8)}{2(0.8)} = 4xy \text{ for } 0 < x < 1 \text{ and } 0 < y < 1.$$

Then, $P(X < 0.5, Y < 0.5 | Z = 0.8) = \int_0^{0.5}\int_0^{0.5} (4xy)dydx = \int_0^{0.5} (x/2)dx = \frac{1}{16} = 0.0625$

5-57. a) $f_{YZ}(y, z) = \int_0^1 (8xyz)dx = 4yz$ for $0 < y < 1$ and $0 < z < 1$.

Then, $f_{X|YZ}(x) = \dfrac{f_{XYZ}(x, y, z)}{f_{YZ}(y, z)} = \dfrac{8x(0.5)(0.8)}{4(0.5)(0.8)} = 2x$ for $0 < x < 1$.

b) Therefore, $P(X < 0.5 | Y = 0.5, Z = 0.8) = \int_0^{0.5} 2xdx = 0.25$

5-58 a) $\iint_{x^2+y^2\le 4}\int_0^4 cdzdydx$ = the volume of a cylinder with a base of radius 2 and a height of 4 = $(\pi 2^2)4 = 16\pi$. Therefore, $c = \dfrac{1}{16\pi}$

b) $P(X^2 + Y^2 \le 1)$ equals the volume of a cylinder of radius $\sqrt{2}$ and a height of 4 ($=8\pi$) times *c*. Therefore, the answer is $\dfrac{8\pi}{16\pi} = 1/2$.

c) P(Z < 2) equals half the volume of the region where $f_{XYZ}(x, y, z)$ is positive times 1/c. Therefore, the answer is 0.5.

d) $E(X) = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^4 \frac{x}{c}dzdydx = \frac{1}{c}\int_{-2}^{2}\left[4xy\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right]dx = \frac{1}{c}\int_{-2}^{2}(8x\sqrt{4-x^2})dx$. Using substitution, $u = 4 - x^2$, $du = -2x\,dx$, and $E(X) = \frac{1}{c}\int 4\sqrt{u}du = \frac{-4}{c}\frac{2}{3}(4-x^2)^{\frac{3}{2}}\Big|_{-2}^{2} = 0$.

5-59. a) $f_{X|1}(x) = \dfrac{f_{XY}(x,1)}{f_Y(1)}$ *and* $f_{XY}(x, y) = \frac{1}{c}\int_0^4 dz = \frac{4}{c} = \frac{1}{4\pi}$ *for* $x^2 + y^2 < 4$.

Also, $f_Y(y) = \frac{1}{c}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^4 dzdx = \frac{8}{c}\sqrt{4-y^2}$ for $-2 < y < 2$.

Then, $f_{X|y}(x) = \dfrac{4/c}{\frac{8}{c}\sqrt{4-y^2}}$ evaluated at y = 1. That is, $f_{X|1}(x) = \frac{1}{2\sqrt{3}}$ for $-\sqrt{3} < x < \sqrt{3}$.

Therefore, $P(X < 1 | Y < 1) = \int_{-\sqrt{3}}^{1} \frac{1}{2\sqrt{3}}dx = \dfrac{1+\sqrt{3}}{2\sqrt{3}} = 0.7887$

b) $f_{XY|1}(x,y)=\frac{f_{XYZ}(x,y,1)}{f_Z(1)}$ and $f_Z(z)=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\frac{1}{c}dydx=\int_{-2}^{2}\frac{2}{c}\sqrt{4-x^2}dx$

Because $f_Z(z)$ is a density over the range 0 < z < 4 that does not depend on Z, $f_Z(z)=1/4$ for

$0<z<4$. Then, $f_{XY|1}(x,y)=\frac{1/c}{1/4}=\frac{1}{4\pi}$ for $x^2+y^2<4$.

Then, $P(x^2+y^2<1|Z=1)=\frac{area\ \ in\ \ x^2+y^2<1}{4\pi}=1/4$.

5-60 $f_{Z|xy}(z)=\frac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)}$ and from part 5-59 a., $f_{XY}(x,y)=\frac{1}{4\pi}$ $\quad for \quad x^2+y^2<4$.

Therefore, $f_{Z|xy}(z)=\frac{\frac{1}{16\pi}}{\frac{1}{4\pi}}=1/4$ for $0<z<4$.

5-61 Determine c such that $f(xyz)=c$ is a joint density probability over the region x>0, y>0 and z>0 with x+y+z<1

$$f(xyz)=c\int_0^1\int_0^{1-x}\int_0^{1-x-y}dzdydx=\int_0^1\int_0^{1-x}c(1-x-y)dydx=\int_0^1\left(c(y-xy-\frac{y^2}{2})\Big|_0^{1-x}\right)dx$$

$$=\int_0^1 c\left((1-x)-x(1-x)-\frac{(1-x)^2}{2}\right)dx=\int_0^1 c\left(\frac{(1-x)^2}{2}\right)dx=c\left(\frac{1}{2}x-\frac{x^2}{2}+\frac{x^3}{6}\right)\Bigg|_0^1$$

$$=c\frac{1}{6}.\quad \text{Therefore, c = 6.}$$

5-62 a.) $P(X<0.5,Y<0.5,Z<0.5)=6\int_0^1\int_0^{1-x}\int_0^{1-x-y}dzdydx\Rightarrow$ The conditions x>0.5, y>0.5, z>0.5 and x+y+z<1 make a space that is a cube with a volume of 0.125. Therefore the probability of $P(X<0.5,Y<0.5,Z<0.5)=6(0.125)=0.75$

b.)

$$P(X<0.5,Y<0.5)=\int_0^{0.5}\int_0^{0.5}6(1-x-y)dydx=\int_0^{0.5}\left(6y-6xy-3y^2\right)\Big|_0^{0.5}dx$$

$$=\int_0^{0.5}\left(\frac{9}{4}-3x\right)dx=\left(\frac{9}{4}x-\frac{3}{2}x^2\right)\Bigg|_0^{0.5}=3/4$$

c.)

$$P(X<0.5)=6\int_0^{0.5}\int_0^{1-x}\int_0^{1-x-y}dzdydx=\int_0^{0.5}\int_0^{1-x}6(1-x-y)dydx=\int_0^{0.5}6(y-xy-\frac{y^2}{2})\Bigg|_0^{1-x}$$

$$=\int_0^{0.5}6(\frac{x^2}{2}-x+\frac{1}{2})dx=\left(x^3-3x^2+3x\right)\Big|_0^{0.5}=0.875$$

d.)

$$E(X) = 6\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} 6x(1-x-y)\,dy\,dx = \int_0^1 6x(y - xy - \frac{y^2}{2})\Big|_0^{1-x}$$

$$= \int_0^1 6(\frac{x^3}{2} - x^2 + \frac{x}{2})dx = \left(\frac{3x^4}{4} - 2x^3 + \frac{3x^2}{2}\right)\Bigg|_0^1 = 0.25$$

5-63 a.)

$$f(x) = 6\int_0^{1-x} \int_0^{1-x-y} dz\,dy = \int_0^{1-x} 6(1-x-y)\,dy = 6\left(y - xy - \frac{y^2}{2}\right)\Bigg|_0^{1-x}$$

$$= 6(\frac{x^2}{2} - x + \frac{1}{2}) = 3(x-1)^2 \text{ for } 0 < x < 1$$

b.)

$$f(x, y) = 6\int_0^{1-x-y} dz = 6(1-x-y)$$

for $x > 0,\ y > 0\ and\ x + y < 1$

c.)

$$f(x \mid y = 0.5, z = 0.5) = \frac{f(x, y = 0.5, z = 0,5)}{f(y = 0.5, z = 0.5)} = \frac{6}{6} = 1 \text{ For, } x = 0$$

d.) The marginal $f_Y(y)$ is similar to $f_X(x)$ and $f_Y(y) = 3(1-y)^2$ for $0 < y < 1$.

$$f_{X|Y}(x \mid 0.5) = \frac{f(x, 0.5)}{f_Y(0.5)} = \frac{6(0.5 - x)}{3(0.25)} = 4(1-2x) \text{ for } x < 0.5$$

5-64 Let X denote the production yield on a day. Then,

$P(X > 1400) = P(Z > \frac{1400-1500}{\sqrt{10000}}) = P(Z > -1) = 0.84134$.

a) Let Y denote the number of days out of five such that the yield exceeds 1400. Then, by independence, Y has a binomial distribution with $n = 5$ and $p =$ 0.8413. Therefore, the answer is $P(Y = 5) = \binom{5}{5}0.8413^5(1-0.8413)^0 = 0.4215$.

b) As in part a., the answer is

$$P(Y \geq 4) = P(Y = 4) + P(Y = 5)$$

$$= \binom{5}{4}0.8413^4(1-0.8413)^1 + 0.4215 = 0.8190$$

5-65. a) Let X denote the weight of a brick. Then,

$P(X > 2.75) = P(Z > \frac{2.75-3}{0.25}) = P(Z > -1) = 0.84134$.

Let Y denote the number of bricks in the sample of 20 that exceed 2.75 pounds. Then, by independence, Y has a binomial distribution with n = 20 and p = 0.84134. Therefore, the answer is $P(Y = 20) = \binom{20}{20} 0.84134^{20} = 0.032$.

b) Let A denote the event that the heaviest brick in the sample exceeds 3.75 pounds. Then, P(A) = 1 - P(A') and A' is the event that all bricks weigh less than 3.75 pounds. As in part a., P(X < 3.75) = P(Z < 3) and

$P(A) = 1 - [P(Z < 3)]^{20} = 1 - 0.99865^{20} = 0.0267$.

5-66 a) Let X denote the grams of luminescent ink. Then,

$P(X < 1.14) = P(Z < \frac{1.14-1.2}{0.3}) = P(Z < -2) = 0.022750$.

Let Y denote the number of bulbs in the sample of 25 that have less than 1.14 grams. Then, by independence, Y has a binomial distribution with n = 25 and p = 0.022750. Therefore, the answer is

$P(Y \geq 1) = 1 - P(Y = 0) = \binom{25}{0} 0.02275^0 (0.97725)^{25} = 1 - 0.5625 = 0.4375$.

b)

$$P(Y \leq 5) = P(Y = 0) + P(Y = 1) + P(Y = 2) + (P(Y = 3) + P(Y = 4) + P(Y = 5)$$
$$= \binom{25}{0} 0.02275^0 (0.97725)^{25} + \binom{25}{1} 0.02275^1 (0.97725)^{24} + \binom{25}{2} 0.02275^2 (0.97725)^{23}$$
$$+ \binom{25}{3} 0.02275^3 (0.97725)^{22} + \binom{25}{4} 0.02275^4 (0.97725)^{21} + \binom{25}{5} 0.02275^5 (0.97725)^{20}.$$
$$= 0.5625 + 0.3274 + 0.09146 + 0.01632 + 0.002090 + 0.0002043 = 0.99997 \cong 1$$

c.) $P(Y = 0) = \binom{25}{0} 0.02275^0 (0.97725)^{25} = 0.5625$

d.) The lamps are normally and independently distributed, therefore, the probabilities can be multiplied.

Section 5-5

5-67. E(X) = 1(3/8)+2(1/2)+4(1/8)=15/8 = 1.875
E(Y) = 3(1/8)+4(1/4)+5(1/2)+6(1/8)=37/8 = 4.625

$$\mathrm{E(XY)} = [1 \times 3 \times (1/8)] + [1 \times 4 \times (1/4)] + [2 \times 5 \times (1/2)] + [4 \times 6 \times (1/8)]$$
$$= 75/8 = 9.375$$
$$\sigma_{XY} = E(XY) - E(X)E(Y) = 9.375 - (1.875)(4.625) = 0.703125$$

$V(X) = 1^2(3/8) + 2^2(1/2) + 4^2(1/8) - (15/8)^2 = 0.8594$
$V(Y) = 3^2(1/8) + 4^2(1/4) + 5^2(1/2) + 6^2(1/8) - (37/8)^2 = 0.7344$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0.703125}{\sqrt{(0.8594)(0.7344)}} = 0.8851$$

5-68 $E(X) = -1(1/8) + (-0.5)(1/4) + 0.5(1/2) + 1(1/8) = 0.125$

$E(Y) = -2(1/8) + (-1)(1/4) + 1(1/2) + 2(1/8) = 0.25$

$\mathrm{E(XY)} = [-1\times -2\times (1/8)] + [-0.5\times -1\times (1/4)] + [0.5\times 1\times (1/2)] + [1\times 2\times (1/8)] = 0.875$

V(X) = 0.4219

V(Y) = 1.6875

$\sigma_{XY} = 0.875 - (0.125)(0.25) = 0.8438$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{0.8438}{\sqrt{0.4219}\sqrt{1.6875}} = 1$$

5-69.

$$\sum_{x=1}^{3}\sum_{y=1}^{3} c(x+y) = 36c, \qquad c = 1/36$$

$$E(X) = \frac{13}{6} \qquad E(Y) = \frac{13}{6} \qquad E(XY) = \frac{14}{3} \qquad \sigma_{xy} = \frac{14}{3} - \left(\frac{13}{6}\right)^2 = \frac{-1}{36}$$

$$E(X^2) = \frac{16}{3} \qquad E(Y^2) = \frac{16}{3} \qquad V(X) = V(Y) = \frac{23}{36}$$

$$\rho = \frac{\frac{-1}{36}}{\sqrt{\frac{23}{36}}\sqrt{\frac{23}{36}}} = -0.0435$$

5-70 $E(X) = 0(0.01) + 1(0.99) = 0.99$

$E(Y) = 0(0.02) + 1(0.98) = 0.98$

$\mathrm{E(XY)} = [0\times 0\times (0.002)] + [0\times 1\times (0.0098)] + [1\times 0\times (0.0198)] + [1\times 1\times (0.9702)] = 0.9702$

V(X) = $0.99 - 0.99^2 = 0.0099$

V(Y) = $0.98 - 0.98^2 = 0.0196$

$\sigma_{XY} = 0.9702 - (0.99)(0.98) = 0$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{0}{\sqrt{0.0099}\sqrt{0.0196}} = 0$$

5-71 $E(X_1) = np_1 = 20(1/3) = 6.67$

$E(X_2) = np_2 = 20(1/3) = 6.67$

$V(X_1) = np_1(1-p_1) = 20(1/3)(2/3) = 4.44$

$V(X_2) = np_2(1-p_2) = 20(1/3)(2/3) = 4.44$

$E(X_1X_2) = n(n-1)p_1p_2 = 20(19)(1/3)(1/3) = 42.22$

$$\sigma_{XY} = 42.22 - 6.67^2 = -2.267 \;\; and \;\; \rho_{XY} = \frac{-2.267}{\sqrt{(4.44)(4.44)}} = -0.51$$

The sign is negative.

5-72 From Exercise 5-40, c=8/81.

From Exercise 5-41, $E(X)$ = 12/5, and $E(Y)$ = 8/5

$$E(XY) = \frac{8}{81}\int_0^3\int_0^x xy(xy)dydx = \frac{8}{81}\int_0^3\int_0^x x^2y^2dydx = \frac{8}{81}\int_0^3 \frac{x^3}{3}x^2dx = \frac{8}{81}\int_0^3 \frac{x^5}{3}dx$$

$$= \left(\frac{8}{81}\right)\left(\frac{3^6}{18}\right) = 4$$

$$\sigma_{xy} = 4 - \left(\frac{12}{5}\right)\left(\frac{8}{5}\right) = 0.16$$

$$E(X^2) = 6 \qquad E(Y^2) = 3$$

$$V(x) = 0.24, \qquad V(Y) = 0.44$$

$$\rho = \frac{0.16}{\sqrt{0.24}\sqrt{0.44}} = 0.4924$$

5-73. Similarly to 5-49, $c = 2/19$

$$E(X) = \frac{2}{19}\int_0^1\int_0^{x+1} xdydx + \frac{2}{19}\int_1^5\int_{x-1}^{x+1} xdydx = 2.614$$

$$E(Y) = \frac{2}{19}\int_0^1\int_0^{x+1} ydydx + \frac{2}{19}\int_1^5\int_{x-1}^{x+1} ydydx = 2.649$$

Now, $E(XY) = \frac{2}{19}\int_0^1\int_0^{x+1} xydydx + \frac{2}{19}\int_1^5\int_{x-1}^{x+1} xydydx = 8.7763$

$$\sigma_{xy} = 8.7763 - (2.614)(2.649) = 1.85181$$

$$E(X^2) = 8.7632 \qquad E(Y^2) = 9.11403$$

$$V(x) = 1.930, \qquad V(Y) = 2.0968$$

$$\rho = \frac{1.852}{\sqrt{1.930}\sqrt{2.062}} = 0.9206$$

5-74

$$E(X)=\int_0^1\int_0^{x+1}\tfrac{x}{7.5}\,dydx+\int_1^4\int_{x-1}^{x+1}\tfrac{x}{7.5}\,dydx$$

$$=\tfrac{1}{7.5}\int_0^1 (x^2+x)dx+\tfrac{2}{7.5}\int_1^4 (x)dx=\tfrac{12}{15}(\tfrac{5}{6})+\tfrac{2}{7.5}(7.5)=\tfrac{19}{9}$$

$E(X^2)$=222,222.2

$V(X)=222222.2-(333.33)^2=111,113.31$

$E(Y^2)$=1,055,556

V(Y)=361,117.11

$$E(XY)=6\times10^{-6}\int_0^\infty\int_x^\infty xye^{-.001x-.002y}dydx=388,888.9$$

$$\sigma_{xy}=388,888.9-(333.33)(833.33)=111,115.01$$

$$\rho=\frac{111,115.01}{\sqrt{111113.31}\sqrt{361117.11}}=0.5547$$

5-75. a) E(X) = 1 E(Y) = 1

$$E(XY)=\int_0^\infty\int_0^\infty xye^{-x-y}dxdy$$

$$=\int_0^\infty xe^{-x}dx\int_0^\infty ye^{-y}dy$$

$$=E(X)E(Y)$$

Therefore, $\sigma_{XY}=\rho_{XY}=0$.

5-76. Suppose the correlation between X and Y is ρ. for constants a, b, c, and d, what is the correlation between the random variables U = aX+b and V = cY+d?

Now, E(U) = a E(X) + b and E(V) = c E(Y) + d.

Also, U - E(U) = a[X - E(X)] and V - E(V) = c[Y - E(Y)]. Then,

$$\sigma_{UV}=E\{[U-E(U)][V-E(V)]\}=acE\{[X-E(X)][Y-E(Y)]\}=ac\sigma_{XY}$$

Also, $\sigma_U^2=E[U-E(U)]^2=a^2E[X-E(X)]^2=a^2\sigma_X^2$ and $\sigma_V^2=c^2\sigma_Y^2$. Then,

$$\rho_{UV}=\frac{ac\sigma_{XY}}{\sqrt{a^2\sigma_X^2}\sqrt{c^2\sigma_Y^2}}=\begin{cases}\rho_{XY} & \text{if a and c are of the same sign}\\ -\rho_{XY} & \text{if a and c differ in sign}\end{cases}$$

5-77 $E(X) = -1(1/4) + 1(1/4) = 0$

$E(Y) = -1(1/4) + 1(1/4) = 0$

$\mathrm{E(XY)} = [-1\times 0\times(1/4)] + [-1\times 0\times(1/4)] + [1\times 0\times(1/4)] + [0\times 1\times(1/4)] = 0$

V(X) = 1/2

V(Y) = 1/2

$\sigma_{XY} = 0 - (0)(0) = 0$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0}{\sqrt{1/2}\sqrt{1/2}} = 0$$

The correlation is zero, but *X* and *Y* are not independent, since, for example, if y=0, X must be –1 or 1.

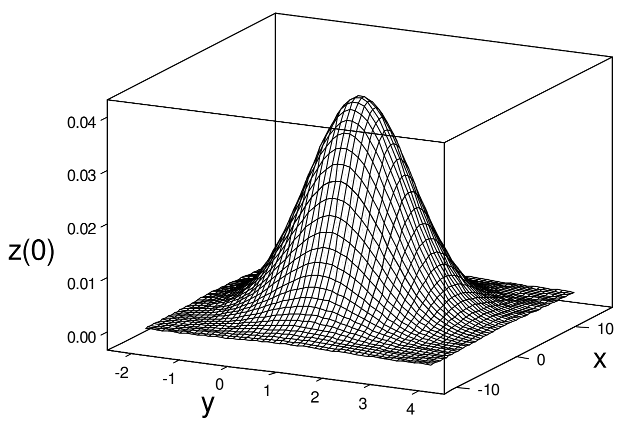
5-78 If X and Y are independent, then $f_{XY}(x, y) = f_X(x) f_Y(y)$ and the range of (X, Y) is rectangular. Therefore,

$$E(XY) = \iint xy f_X(x) f_Y(y) dxdy = \int x f_X(x) dx \int y f_Y(y) dy = E(X)E(Y)$$

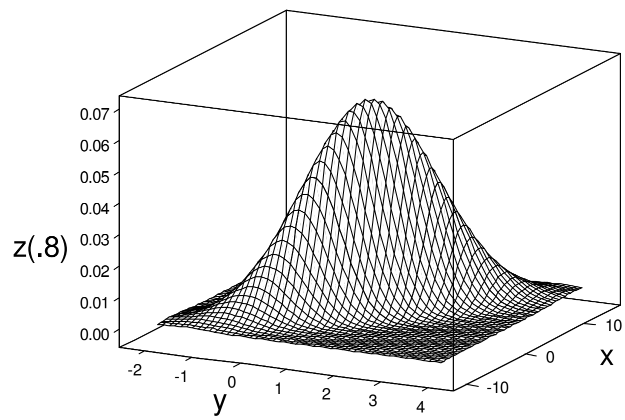
hence $\sigma_{XY}=0$

Section 5-6

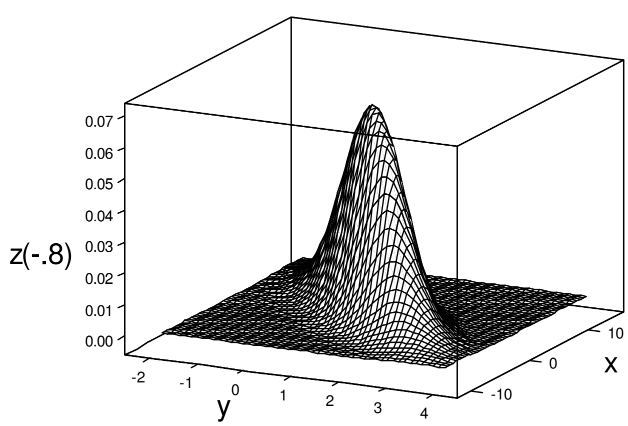
5-79 a.)



b.)


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
z(.8)
-2
-1
0
1
2
3
4
y
10
0
-10
x

c)


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
z(-.8)
-2
-1
0
1
2
3
4
y
10
0
-10
x

5-80 Because $\rho = 0$ and X and Y are normally distributed, X and Y are independent. Therefore,

P(2.95 < X < 3.05, 7.60 < Y < 7.80) = P(2.95 < X < 3.05) P(7.60 < Y < 7.80) =

$P(\frac{2.95-3}{0.04} < Z < \frac{3.05-3}{0.04})P(\frac{7.60-7.70}{0.08} < Z < \frac{7.80-7.70}{0.08}) = 0.7887^2 = 0.6220$

5-81. Because $\rho = 0$ and X and Y are normally distributed, X and Y are independent. Therefore,

μ_X = 0.1mm σ_X=0.00031mm μ_Y = 0.23mm σ_Y=0.00017mm

Probability X is within specification limits is

$$P(0.099535 < X < 0.100465) = P\left(\frac{0.099535-0.1}{0.00031} < Z < \frac{0.100465-0.1}{0.00031}\right)$$

$$= P(-1.5 < Z < 1.5) = P(Z < 1.5) - P(Z < -1.5) = 0.8664$$

Probability that Y is within specification limits is

$$P(0.22966 < X < 0.23034) = P\left(\frac{0.22966-0.23}{0.00017} < Z < \frac{0.23034-0.23}{0.00017}\right)$$

$$= P(-2 < Z < 2) = P(Z < 2) - P(Z < -2) = 0.9545$$

Probability that a randomly selected lamp is within specification limits is (0.8664)(.9594)=0.8270

5-82 a) By completing the square in the numerator of the exponent of the bivariate normal PDF, the joint PDF can be written as

$$f_{Y|X=x} = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{\left[\frac{1}{\sigma_Y^2}\left[(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2+(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}{2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{\left[\frac{x-\mu_x}{\sigma_x}\right]^2}{2}}}$$

Also, $f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{\left[\frac{x-\mu_x}{\sigma_x}\right]^2}{2}}$ By definition,

$$f_{Y|X=x} = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{\left[\frac{1}{\sigma_Y^2}\left[(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2+(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}{2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{\left[\frac{x-\mu_x}{\sigma_x}\right]^2}{2(1-\rho^2)}}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}}e^{-\frac{\frac{1}{\sigma_Y^2}\left[(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2}{2(1-\rho^2)}}$$

Now $f_{Y|X=x}$ is in the form of a normal distribution.

b) E(Y|X=x) = $\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)$. This answer can be seen from part 5-82a. Since the PDF is in the form of a normal distribution, then the mean can be obtained from the exponent.

c) V(Y|X=x) = $\sigma_y^2(1-\rho^2)$. This answer can be seen from part 5-82a. Since the PDF is in the form of a normal distribution, then the variance can be obtained from the exponent.

5-83

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X{}^2}+\frac{(y-\mu_Y)^2}{\sigma_Y{}^2}\right]}\right]dxdy =$$

$$\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X{}^2}\right]}\right]dx\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left[\frac{(y-\mu_Y)^2}{\sigma_Y{}^2}\right]}\right]dy$$

and each of the last two integrals is recognized as the integral of a normal probability density function from $-\infty$ to ∞. That is, each integral equals one. Since $f_{XY}(x,y) = f(x)f(y)$ then *X* and *Y* are independent.

5-84 Let $f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho(X-\mu_X)(Y-\mu_X)}{\sigma_X\sigma_Y} + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]}{2(1-\rho^2)}}$

Completing the square in the numerator of the exponent we get:

$$\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho(X-\mu_X)(Y-\mu_X)}{\sigma_X\sigma_Y} + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right] = \left[\left(\frac{Y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{X-\mu_X}{\sigma_X}\right)\right]^2 + (1-\rho^2)\left(\frac{X-\mu_X}{\sigma_X}\right)^2$$

But,

$$\left(\frac{Y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{X-\mu_X}{\sigma_X}\right) = \frac{1}{\sigma_Y}\left[(Y-\mu_Y) - \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_x)\right] = \frac{1}{\sigma_Y}\left[(Y-(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_x))\right]$$

Substituting into $f_{XY}(x,y)$, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\frac{1}{\sigma_Y^2}\left[y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2 + (1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}{2(1-\rho^2)}} dydx$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \times \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\left[\frac{\left(y-(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x))\right)^2}{2\sigma_x^2(1-\rho^2)}\right]} dy$$

The integrand in the second integral above is in the form of a normally distributed random variable. By definition of the integral over this function, the second integral is equal to 1:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\left[\frac{\left(y-(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x))\right)^2}{2\sigma_x^2(1-\rho^2)}\right]} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \times 1$$

The remaining integral is also the integral of a normally distributed random variable and therefore, it also integrates to 1, by definition. Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = 1$$

5-85

$$f_X(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} \right] dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-0.5}{1-\rho^2}\frac{(x-\mu_X)^2}{\sigma_X^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2}\left\{\left[\frac{(y-\mu_Y)}{\sigma_Y} - \frac{\rho(x-\mu_X)}{\sigma_X}\right]^2 - \left[\frac{\rho(x-\mu_X)}{\sigma_X}\right]^2\right\}} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-0.5\frac{(x-\mu_X)^2}{\sigma_X^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2}\left[\frac{(y-\mu_Y)}{\sigma_Y} - \frac{\rho(x-\mu_X)}{\sigma_X}\right]^2} dy$$

The last integral is recognized as the integral of a normal probability density with mean $\mu_Y + \frac{\sigma_Y\rho(x-\mu_X)}{\sigma_X}$ and variance $\sigma_Y{}^2(1-\rho^2)$. Therefore, the last integral equals one and the requested result is obtained.

5-86 $E(X)=\mu_X, E(Y)=\mu_Y, V(X)=\sigma_X^2,\ \ and\ \ V(Y)=\sigma_Y^2$. Also,

$$E(X-\mu_X)(Y-\mu_Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x-\mu_X)(y-\mu_Y)e^{\frac{-0.5}{1-\rho^2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}-\frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}}{2\pi\sigma_X\sigma_Y(1-\rho^2)^{1/2}}dxdy$$

Let $u=\frac{x-\mu_X}{\sigma_X}\ \ and\ \ v=\frac{y-\mu_Y}{\sigma_Y}$. Then,

$$E(X-\mu_X)(Y-\mu_Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{uve^{\frac{-0.5}{1-\rho^2}\left[u^2-2\rho uv+v^2\right]}}{2\pi(1-\rho^2)^{1/2}}\ \sigma_X\sigma_Y dudv$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{uve^{\frac{-0.5}{1-\rho^2}\left\{[u-\rho v]^2+(1-\rho^2)v^2\right\}}}{2\pi(1-\rho^2)^{1/2}}\ \sigma_X\sigma_Y dudv$$

The integral with respect to u is recognized as a constant times the mean of a normal random variable with mean ρv and variance $1-\rho^2$. Therefore,

$$E(X-\mu_X)(Y-\mu_Y)=\int_{-\infty}^{\infty}\frac{v}{\sqrt{2\pi}}\ e^{-0.5v^2}\rho v\sigma_X\sigma_Y dv=\rho\sigma_X\sigma_Y\int_{-\infty}^{\infty}\frac{v^2}{\sqrt{2\pi}}\ e^{-0.5v^2}dv\,.$$

The last integral is recognized as the variance of a normal random variable with mean 0 and variance 1. Therefore, $E(X-\mu_X)(Y-\mu_Y)=\rho\sigma_X\sigma_Y$ and the correlation between X and Y is ρ.

Section 5-7

5-87. a) E(2X + 3Y) = 2(0) + 3(10) = 30
b) V(2X + 3Y) = 4V(X) + 9V(Y) = 97
c) 2X + 3Y is normally distributed with mean 30 and variance 97. Therefore,
$P(2X+3Y<30)=P(Z<\frac{30-30}{\sqrt{97}})=P(Z<0)=0.5$
d) $P(2X+3Y<40)=P(Z<\frac{40-30}{\sqrt{97}})=P(Z<1.02)=0.8461$

5-88 Y = 10X and E(Y) =10E(X) = 50mm.
$V(Y)=10^2V(X)=25mm^2$

5-89 a) Let T denote the total thickness. Then, T = X + Y and E(T) = 4 mm,

V(T) = $0.1^2 + 0.1^2 = 0.02 mm^2$, and $\sigma_T = 0.1414$ mm.

b)

$$P(T > 4.3) = P\left(Z > \frac{4.3-4}{0.1414}\right) = P(Z > 2.12)$$
$$= 1 - P(Z < 2.12) = 1 - 0.983 = 0.0170$$

5-90 a) X~N(0.1, 0.00031) and Y~N(0.23, 0.00017) Let T denote the total thickness. Then, T = X + Y and E(T) = 0.33 mm,

V(T) = $0.00031^2 + 0.00017^2 = 1.25x10^{-7} mm^2$, and $\sigma_T = 0.000354$ mm.

$$P(T < 0.2337) = P\left(Z < \frac{0.2337 - 0.33}{0.000354}\right) = P(Z < -272) \cong 0$$

b)

$$P(T > 0.2405) = P\left(Z > \frac{0.2405 - 0.33}{0.000345}\right) = P(Z > -253) = 1 - P(Z < 253) \cong 1$$

5-91. Let D denote the width of the casing minus the width of the door. Then, D is normally distributed.

a) E(D) = 1/8 V(D) = $(\frac{1}{8})^2 + (\frac{1}{16})^2 = \frac{5}{256}$

b) $P(D > \frac{1}{4}) = P(Z > \frac{\frac{1}{4}-\frac{1}{8}}{\sqrt{5/256}}) = P(Z > 0.89) = 0.187$

c) $P(D < 0) = P(Z < \frac{0-\frac{1}{8}}{\sqrt{5/256}}) = P(Z < -0.89) = 0.187$

5-92 D = A - B - C

a) E(D) = 10 - 2 - 2 = 6 mm

$V(D) = 0.1^2 + 0.05^2 + 0.05^2 = 0.015mm^2$

$\sigma_D = 0.1225mm$

b) P(D < 5.9) = P(Z < $\frac{5.9-6}{0.1225}$) = P(Z < -0.82) = 0.206.

5-93. a) Let $\overline{X}$ denote the average fill-volume of 100 cans. $\sigma_{\overline{X}} = \sqrt{0.5^2/100} = 0.05$.

b) E($\overline{X}$) = 12.1 and $P(\overline{X} < 12) = P\left(Z < \frac{12-12.1}{0.05}\right) = P(Z < -2) = 0.023$

c) P($\overline{X}$ < 12) = 0.005 implies that $P\left(Z < \frac{12-\mu}{0.05}\right) = 0.005.$

Then $\frac{12-\mu}{0.05}$ = -2.58 and $\mu = 12.129$.

d.) P($\overline{X}$ < 12) = 0.005 implies that $P\left(Z < \frac{12-12.1}{\sigma/\sqrt{100}}\right) = 0.005.$

Then $\frac{12-12.1}{\sigma/\sqrt{100}}$ = -2.58 and $\sigma = 0.388$.

e.) P($\overline{X}$ < 12) = 0.01 implies that $P\left(Z < \dfrac{12-12.1}{0.5/\sqrt{n}}\right) = 0.01$.

Then $\frac{12-12.1}{0.5/\sqrt{n}}$ = -2.33 and $n = 135.72 \cong 136$.

5-94 Let $\overline{X}$ denote the average thickness of 10 wafers. Then, E($\overline{X}$) = 10 and V($\overline{X}$) = 0.1.

a) $P(9 < \overline{X} < 11) = P(\frac{9-10}{\sqrt{0.1}} < Z < \frac{11-10}{\sqrt{0.1}}) = P(-3.16 < Z < 3.16) = 0.998$.

The answer is 1 − 0.998 = 0.002

b) $P(\overline{X} > 11) = 0.01$ *and* $\sigma_{\overline{X}} = \frac{1}{\sqrt{n}}$.

Therefore, $P(\overline{X} > 11) = P(Z > \frac{11-10}{1/\sqrt{n}}) = 0.01$, $\frac{11-10}{1/\sqrt{n}}$ = 2.33 and n = 5.43 which is rounded up to 6.

c.) $P(\overline{X} > 11) = 0.0005$ *and* $\sigma_{\overline{X}} = \sigma/\sqrt{10}$.

Therefore, $P(\overline{X} > 11) = P(Z > \frac{11-10}{\sigma/\sqrt{10}}) = 0.0005$, $\frac{11-10}{\sigma/\sqrt{10}}$ = 3.29

$\sigma = \sqrt{10}/3.29 = 0.9612$

5-95. X~N(160, 900)

a) Let Y = 25X, E(Y)=25E(X)=4000, V(Y) = 25^2(900)=562500

P(Y>4300) =

$$P\left(Z > \frac{4300-4000}{\sqrt{562500}}\right) = P(Z > 0.4) = 1 - P(Z < 0.4) = 1 - 0.6554 = 0.3446$$

b.) c) P(Y > x) = 0.0001 implies that $P\left(Z > \dfrac{x-4000}{\sqrt{562500}}\right) = 0.0001$.

Then $\frac{x-4000}{750}$ = 3.72 and $x = 6790$.

Supplemental Exercises

5-96 The sum of $\sum_x \sum_y f(x, y) = 1$, $\left(\frac{1}{4}\right)+\left(\frac{1}{8}\right)+\left(\frac{1}{8}\right)+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)=1$

and $f_{XY}(x, y) \geq 0$

5-97. a) $P(X < 0.5, Y < 1.5) = f_{XY}(0,1) + f_{XY}(0,0) = 1/8 + 1/4 = 3/8$.

b) $P(X \leq 1) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1) = 3/4$

c) $P(Y < 1.5) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1) = 3/4$

d) $P(X > 0.5, Y < 1.5) = f_{XY}(1,0) + f_{XY}(1,1) = 3/8$

e) E(X) = 0(3/8) + 1(3/8) + 2(1/4) = 7/8.

V(X) = 0^2(3/8) + 1^2(3/8) + 2^2(1/4) - $7/8^2$ =39/64

E(Y) = 1(3/8) + 0(3/8) + 2(1/4) = 7/8.

. V(Y) = 1^2(3/8) + 0^2(3/8) + 2^2(1/4) - $7/8^2$ =39/64

5-98 a) $f_X(x) = \sum_y f_{XY}(x,y)$ *and* $f_X(0) = 3/8,\ f_X(1) = 3/8,\ f_X(2) = 1/4$.

b) $f_{Y|1}(y) = \dfrac{f_{XY}(1,y)}{f_X(1)}$ *and* $f_{Y|1}(0) = \frac{1/8}{3/8} = 1/3,\ f_{Y|1}(1) = \frac{1/4}{3/8} = 2/3$.

c) $E(Y \mid X = 1) = \sum_{x=1} y f_{XY}(1,y) = 0(1/3) + 1(2/3) = 2/3$

d) Because the range of (X, Y) is not rectangular, X and Y are not independent.

e.) E(XY) = 1.25, E(X) = E(Y)= 0.875 V(X) = V(Y) = 0.6094
COV(X,Y)=E(XY)-E(X)E(Y)= 1.25-0.875^2=0.4844

$$\rho_{XY} = \frac{0.4844}{\sqrt{0.6094}\sqrt{0.6094}} = 0.7949$$

5-99 a.) $P(X = 2, Y = 4, Z = 14) = \dfrac{20!}{2!4!14!} 0.10^2 0.20^4 0.70^{14} = 0.0631$

b.) $P(X = 0) = 0.10^0 0.90^{20} = 0.1216$

c.) $E(X) = np_1 = 20(0.10) = 2$

$V(X) = np_1(1 - p_1) = 20(0.10)(0.9) = 1.8$

d.) $f_{X|Z=z}(X \mid Z = 19)\, \dfrac{f_{XZ}(x,z)}{f_Z(z)}$

$$f_{XZ}(xz) = \frac{20!}{x!z!(20-x-z)!} 0.1^x 0.2^{20-x-z} 0.7^z$$

$$f_Z(z) = \frac{20!}{z!(20-z)!} 0.3^{20-z} 0.7^z$$

$$f_{X|Z=z}(X \mid Z = 19)\, \frac{f_{XZ}(x,z)}{f_Z(z)} = \frac{(20-z)!}{x!(20-x-z)!} \frac{0.1^x 0.2^{20-x-z}}{0.3^{20-z}} = \frac{(20-z)!}{x!(20-x-z)!} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{20-x-z}$$

Therefore, X is a binomial random variable with n=20-z and p=1/3. When z=19,

$f_{X|19}(0) = \dfrac{2}{3}$ and $f_{X|19}(1) = \dfrac{1}{3}$.

e.) $E(X \mid Z = 19) = 0\left(\dfrac{2}{3}\right) + 1\left(\dfrac{1}{3}\right) = \dfrac{1}{3}$

5-100 Let X, Y, and Z denote the number of bolts rated high, moderate, and low. Then, X, Y, and Z have a multinomial distribution.

a) $P(X = 12, Y = 6, Z = 2) = \dfrac{20!}{12!6!2!} 0.6^{12} 0.3^6 0.1^2 = 0.0560$.

b) Because X, Y, and Z are multinomial, the marginal distribution of Z is binomial with n = 20 and p = 0.1.

c) E(Z) = np = 20(0.1) = 2.

5-101. a) $f_{Z|16}(z) = \dfrac{f_{XZ}(16,z)}{f_X(16)}$ and $f_{XZ}(x,z) = \dfrac{20!}{x!z!(20-x-z)!}0.6^x 0.3^{(20-x-z)} 0.1^z$ for

$x + z \le 20$ *and* $0 \le x, 0 \le z$. Then,

$$f_{Z|16}(z) = \frac{\frac{20!}{16!z!(4-z)!}0.6^{16}0.3^{(4-z)}0.1^z}{\frac{20!}{16!4!}0.6^{16}0.4^4} = \frac{4!}{z!(4-z)!}\left(\frac{0.3}{0.4}\right)^{4-z}\left(\frac{0.1}{0.4}\right)^z$$

for $0 \le z \le 4$. That is the distribution of Z given X = 16 is binomial with n = 4 and p = 0.25.

b) From part a., E(Z) = 4 (0.25) = 1.

c) Because the conditional distribution of Z given X = 16 does not equal the marginal distribution of Z, X
and Z are not independent.

5-102 Let X, Y, and Z denote the number of calls answered in two rings or less, three or four rings, and five rings or more, respectively.

a) $P(X = 8, Y = 1, Z = 1) = \dfrac{10!}{8!1!1!}0.7^8 0.25^1 0.05^1 = 0.0649$

b) Let W denote the number of calls answered in four rings or less. Then, W is a binomial random variable with n = 10 and p = 0.95.

Therefore, P(W = 10) = $\binom{10}{10}0.95^{10}0.05^0 = 0.5987$.

c) E(W) = 10(0.95) = 9.5.

5-103 a) $f_{Z|8}(z) = \dfrac{f_{XZ}(8,z)}{f_X(8)}$ and $f_{XZ}(x,z) = \dfrac{10!}{x!z!(10-x-z)!}0.70^x 0.25^{(10-x-z)} 0.05^z$ for

$x + z \le 10$ *and* $0 \le x, 0 \le z$. Then,

$$f_{Z|8}(z) = \frac{\frac{10!}{8!z!(2-z)!}0.70^8 0.25^{(2-z)}0.05^z}{\frac{10!}{8!2!}0.70^8 0.30^2} = \frac{2!}{z!(2-z)!}\left(\frac{0.25}{0.30}\right)^{2-z}\left(\frac{0.05}{0.30}\right)^z$$

for $0 \le z \le 2$. That is Z is binomial with n =2 and p = 0.05/0.30 = 1/6.

b) E(Z) given X = 8 is 2(1/6) = 1/3.

c) Because the conditional distribution of Z given X = 8 does not equal the marginal distribution of Z, X and Z are not independent.

5-104 $\displaystyle\int_0^3\int_0^2 cx^2 y\,dy\,dx = \int_0^3 cx^2 \left.\frac{y^2}{2}\right|_0^2 dx = 2c\left.\frac{x^3}{3}\right|_0^3 = 18c$. Therefore, c = 1/18.

5-105. a) $P(X<1,Y<1)=\int_0^1\int_0^1 \frac{1}{18}x^2 y\,dy\,dx=\int_0^1 \frac{1}{18}x^2 \left.\frac{y^2}{2}\right|_0^1 dx=\left.\frac{1}{36}\frac{x^3}{3}\right|_0^1=\frac{1}{108}$

b) $P(X<2.5)=\int_0^{2.5}\int_0^2 \frac{1}{18}x^2 y\,dy\,dx=\int_0^{2.5} \frac{1}{18}x^2 \left.\frac{y^2}{2}\right|_0^2 dx=\left.\frac{1}{9}\frac{x^3}{3}\right|_0^{2.5}=0.5787$

c) $P(1<Y<2.5)=\int_0^3\int_1^2 \frac{1}{18}x^2 y\,dy\,dx=\int_0^3 \frac{1}{18}x^2 \left.\frac{y^2}{2}\right|_1^2 dx=\left.\frac{1}{12}\frac{x^3}{3}\right|_0^3=\frac{3}{4}$

d)

$$P(X>2,1<Y<1.5)=\int_2^3\int_1^{1.5} \frac{1}{18}x^2 y\,dy\,dx=\int_2^3 \frac{1}{18}x^2 \left.\frac{y^2}{2}\right|_1^{1.5} dx=\left.\frac{5}{144}\frac{x^3}{3}\right|_2^3$$

$$=\frac{95}{432}=0.2199$$

e) $E(X)=\int_0^3\int_0^2 \frac{1}{18}x^3 y\,dy\,dx=\int_0^3 \frac{1}{18}x^3 2\,dx=\left.\frac{1}{9}\frac{x^4}{4}\right|_0^3=\frac{9}{4}$

f) $E(Y)=\int_0^3\int_0^2 \frac{1}{18}x^2 y^2\,dy\,dx=\int_0^3 \frac{1}{18}x^2 \frac{8}{3}\,dx=\left.\frac{4}{27}\frac{x^3}{3}\right|_0^3=\frac{4}{3}$

5-106 a) $f_X(x)=\int_0^2 \frac{1}{18}x^2 y\,dy=\frac{1}{9}x^2 \;\; for\; 0<x<3$

b) $f_{Y|X}(y)=\dfrac{f_{XY}(1,y)}{f_X(1)}=\dfrac{\frac{1}{18}y}{\frac{1}{9}}=\dfrac{y}{2}$ for 0 < y < 2.

c) $f_{X|1}(x)=\dfrac{f_{XY}(x,1)}{f_Y(1)}=\dfrac{\frac{1}{18}x^2}{f_Y(1)}$ and $f_Y(y)=\int_0^3 \frac{1}{18}x^2 y\,dx=\frac{y}{2} \;\; for\; 0<y<2$.

Therefore, $f_{X|1}(x)=\dfrac{\frac{1}{18}x^2}{1/2}=\frac{1}{9}x^2$ for 0 < x < 3.

5-107. The region $x^2+y^2\le 1$ and 0 < z < 4 is a cylinder of radius 1 (and base area π) and height 4. Therefore, the volume of the cylinder is 4π and $f_{XYZ}(x,y,z)=\frac{1}{4\pi}$ for $x^2+y^2\le 1$ and 0 < z < 4.

a) The region $X^2+Y^2\le 0.5$ is a cylinder of radius $\sqrt{0.5}$ and height 4. Therefore,
$P(X^2+Y^2\le 0.5)=\frac{4(0.5\pi)}{4\pi}=1/2$.

b) The region $X^2+Y^2\le 0.5$ and 0 < z < 2 is a cylinder of radius $\sqrt{0.5}$ and height 2. Therefore,
$P(X^2+Y^2\le 0.5, Z<2)=\frac{2(0.5\pi)}{4\pi}=1/4$

c) $f_{XY|1}(x,y) = \dfrac{f_{XYZ}(x,y,1)}{f_Z(1)}$ and $f_Z(z) = \iint\limits_{x^2+y^2\le 1} \frac{1}{4\pi} dydx = 1/4$

for 0 < z < 4. Then, $f_{XY|1}(x,y) = \dfrac{1/4\pi}{1/4} = \frac{1}{\pi} \;\; for\; x^2 + y^2 \le 1$.

d) $f_X(x) = \int\limits_0^4 \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4\pi} dydz = \int\limits_0^4 \frac{1}{2\pi}\sqrt{1-x^2}\, dz = \frac{2}{\pi}\sqrt{1-x^2}$ for -1 < x < 1

5-108 a) $f_{Z|0,0}(z) = \dfrac{f_{XYZ}(0,0,z)}{f_{XY}(0,0)}$ and $f_{XY}(x,y) = \int\limits_0^4 \frac{1}{4\pi} dz = 1/\pi$ for $x^2 + y^2 \le 1$. Then,

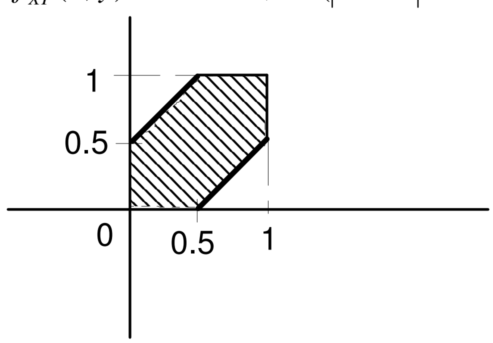
$f_{Z|0,0}(z) = \dfrac{1/4\pi}{1/\pi} = 1/4$ for 0 < z < 4 and $\mu_{Z|0,0} = 2$.

b) $f_{Z|xy}(z) = \dfrac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)} = \dfrac{1/4\pi}{1/\pi} = 1/4$ for 0 < z < 4. Then, E(Z) given X = x and Y = y is

$\int\limits_0^4 \frac{z}{4} dz = 2$.

5-109. $f_{XY}(x,y) = c$ for 0 < x < 1 and 0 < y < 1. Then, $\int\limits_0^1 \int\limits_0^1 c\,dxdy = 1$ and c = 1. Because

$f_{XY}(x,y)$ is constant, $P(|X - Y| < 0.5)$ is the area of the shaded region below



That is, $P(|X - Y| < 0.5)$ = 3/4.

5-110 a) Let $X_1, X_2, \ldots, X_6$ denote the lifetimes of the six components, respectively. Because of independence,

$P(X_1 > 5000, X_2 > 5000, \ldots, X_6 > 5000) = P(X_1 > 5000)P(X_2 > 5000)\ldots P(X_6 > 5000)$

If X is exponentially distributed with mean θ, then $\lambda = \frac{1}{\theta}$ and

$$P(X > x) = \int\limits_x^\infty \frac{1}{\theta}\; e^{-t/\theta} dt = -e^{-t/\theta}\Big|_x^\infty = e^{-x/\theta}$$. Therefore, the answer is

$e^{-5/8} e^{-0.5} e^{-0.5} e^{-0.25} e^{-0.25} e^{-0.2} = e^{-2.325} = 0.0978$.

b) The probability that at least one component lifetime exceeds 25,000 hours is the same as 1 minus the probability that none of the component lifetimes exceed 25,000 hours. Thus,

$1-P(X_a<25{,}000, X_2<25{,}000, \ldots, X_6<25{,}000)=1-P(X_1<25{,}000)\ldots P(X_6<25{,}000)$
$=1-(1-e^{-25/8})(1-e^{-2.5})(1-e^{-2.5})(1-e^{-1.25})(1-e^{-1.25})(1-e^{-1})=1-.2592=0.7408$

5-111. Let X, Y, and Z denote the number of problems that result in functional, minor, and no defects, respectively.

a) $P(X=2, Y=5)=P(X=2, Y=5, Z=3)=\frac{10!}{2!5!3!}0.2^2 0.5^5 0.3^3=0.085$

b) Z is binomial with n = 10 and p = 0.3.

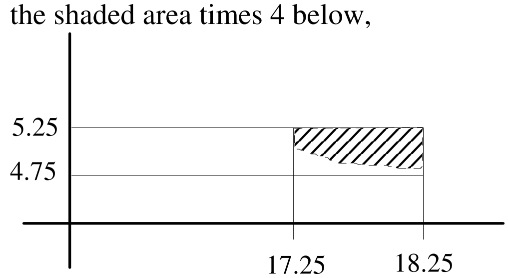
c) E(Z) = 10(0.3) = 3.

5-112 a) Let $\overline{X}$ denote the mean weight of the 25 bricks in the sample. Then, E($\overline{X}$) = 3 and $\sigma_{\overline{X}}=\frac{0.25}{\sqrt{25}}=0.05$. Then, P($\overline{X}$ < 2.95) = P(Z < $\frac{2.95-3}{0.05}$) = P (Z < -1) = 0.159.

b) $P(\overline{X}>x)=P(Z>\frac{x-3}{.05})=0.99$. So, $\frac{x-3}{0.05}=$ -2.33 and x=2.8835.

5-113. a.)

Because $\int_{17.75}^{18.25}\int_{4.75}^{5.25} c\,dy\,dx=0.25c$, c = 4. The area of a panel is XY and P(XY > 90) is the shaded area times 4 below,



That is, $\int_{17.75}^{18.25}\int_{90/x}^{5.25} 4\,dy\,dx=4\int_{17.75}^{18.25} 5.25-\frac{90}{x}\,dx=4(5.25x-90\ln x\Big|_{17.75}^{18.25})=0.499$

b. The perimeter of a panel is 2X+2Y and we want P(2X+2Y >46)

$$\int_{17.75}^{18.25}\int_{23-x}^{5.25} 4\,dy\,dx=4\int_{17.75}^{18.25} 5.25-(23-x)\,dx$$

$$=4\int_{17.75}^{18.25}(-17.75+x)\,dx=4(-17.75x+\frac{x^2}{2}\Big|_{17.75}^{18.25})=0.5$$

5-114 a)Let X denote the weight of a piece of candy and X~N(0.1, 0.01). Each package has 16 candies, then P is the total weight of the package with 16 pieces and E(P) = 16(0.1)=1.6 ounces and V(P) = $16^2(0.01^2)$=0.0256 ounces^2

b) $P(P < 1.6) = P(Z < \frac{1.6-1.6}{0.16}) = P(Z < 0) = 0.5$.

c) Let Y equal the total weight of the package with 17 pieces, E(Y) = 17(0.1)=1.7 ounces and V(Y) = $17^2(0.01^2)$=0.0289 ounces^2

$P(Y < 1.6) = P(Z < \frac{1.6-1.7}{\sqrt{0.0289}}) = P(Z < -0.59) = 0.2776$.

5-115. Let $\overline{X}$ denote the average time to locate 10 parts. Then, E($\overline{X}$) =45 and $\sigma_{\overline{X}} = \frac{30}{\sqrt{10}}$

a) $P(\overline{X} > 60) = P(Z > \frac{60-45}{30/\sqrt{10}}) = P(Z > 1.58) = 0.057$

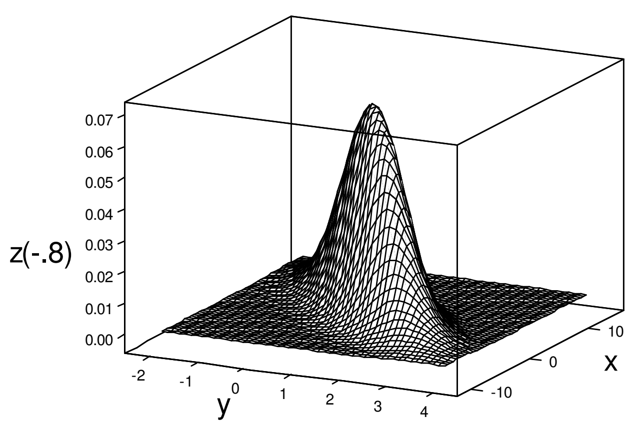
b) Let Y denote the total time to locate 10 parts. Then, Y > 600 if and only if $\overline{X}$ > 60. Therefore, the answer is the same as part a.

5-116 a) Let Y denote the weight of an assembly. Then, E(Y) = 4 + 5.5 + 10 + 8 = 27.5 and V(Y)=$0.4^2 + 0.5^2 + 0.2^2 + 0.5^2 = 0.7$.

$P(Y > 29.5) = P(Z > \frac{29.5-27.5}{\sqrt{0.7}}) = P(Z > 2.39) = 0.0084$

b) Let $\overline{X}$ denote the mean weight of 8 independent assemblies. Then, E($\overline{X}$) = 27.5 and V($\overline{X}$) = 0.7/8 = 0.0875. Also, $P(\overline{X} > 29) = P(Z > \frac{29-27.5}{\sqrt{0.0875}}) = P(Z > 5.07) = 0$.

5-117



5-118

$$f_{XY}(x,y)=\frac{1}{1.2\pi}e^{\left[\frac{-1}{0.72}\{(x-1)^2-1.6(x-1)(y-2)+(y-2)^2\}\right]}$$

$$f_{XY}(x,y)=\frac{1}{2\pi\sqrt{.36}}e^{\left[\frac{-1}{2(0.36)}\{(x-1)^2-1.6(x-1)(y-2)+(y-2)^2\}\right]}$$

$$f_{XY}(x,y)=\frac{1}{2\pi\sqrt{1-.8^2}}e^{\left[\frac{-1}{2(1-.8^2)}\{(x-1)^2-2(.8)(x-1)(y-2)+(y-2)^2\}\right]}$$

$E(X)=1,\ E(Y)=2\ \ V(X)=1\ V(Y)=1$ and $\rho = 0.8$

5-119 Let T denote the total thickness. Then, $T = X_1 + X_2$ and

a.) E(T) = 0.5+1=1.5 mm

$V(T)=V(X_1)+V(X_2)+2Cov(X_1X_2)=0.01+0.04+2(0.014)=0.078mm^2$

where $Cov(XY)=\rho\sigma_X\sigma_Y=0.7(0.1)(0.2)=0.014$

b.) $P(T<1)=P\left(Z<\frac{1-1.5}{\sqrt{0.078}}\right)=P(Z<-1.79)=0.0367$

c.) Let P denote the total thickness. Then, $P = 2X_1 +3 X_2$ and

E(P) =2(0.5)+3(1)=4 mm

$V(P)=4V(X_1)+9V(X_2)+$

$2(2)(3)Cov(X_1X_2)=4(0.01)+9(0.04)+2(2)(3)(0.014)=0.568mm^2$

where $Cov(XY)=\rho\sigma_X\sigma_Y=0.7(0.1)(0.2)=0.014$

5-120 Let T denote the total thickness. Then, $T = X_1 + X_2 + X_3$ and

a) E(T) = 0.5+1+1.5 =3 mm

$V(T)=V(X_1)+V(X_2)+V(X_3)+2Cov(X_1X_2)+2Cov(X_2X_3)+$

$2Cov(X_1X_3)=0.01+0.04+0.09+2(0.014)+2(0.03)+2(0.009)=0.246mm^2$

where $Cov(XY)=\rho\sigma_X\sigma_Y$

b.) $P(T<1.5)=P\left(Z<\frac{1.5-3}{0.246}\right)=P(Z<-6.10)\cong 0$

5-121 Let X and Y denote the percentage returns for security one and two respectively.

If ½ of the total dollars is invested in each then ½X+ ½Y is the percentage return.

E(½X+ ½Y)= 0.05 (or 5 if given in terms of percent)

V(½X+ ½Y)=1/4 V(X)+1/4V(Y)+2(1/2)(1/2)Cov(X,Y)

where Cov(XY)=$\rho\sigma_X\sigma_Y$=-0.5(2)(4)=-4

V(½X+ ½Y)=1/4(4)+1/4(6)-2=3

Also, E(X)=5 and V(X) = 4. Therefore, the strategy that splits between the securities has a lower standard deviation of percentage return than investing 2million in the first security.

Mind-Expanding Exercises

5-122. By the independence,

$$P(X_1 \in A_1, X_2 \in A_2, ..., X_p \in A_p) = \int_{A_1} \int_{A_2} ... \int_{A_p} f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_p}(x_p) dx_1 dx_2 ... dx_p$$

$$= \left[\int_{A_1} f_{X_1}(x_1) dx_1 \right] \left[\int_{A_2} f_{X_2}(x_2) dx_2 \right] ... \left[\int_{A_p} f_{X_p}(x_p) dx_p \right]$$

$$= P(X_1 \in A_1) P(X_2 \in A_2) ... P(X_p \in A_p)$$

5-123 $E(Y) = c_1\mu_1 + c_2\mu_2 + ... + c_p\mu_p.$

Also,

$$V(Y) = \int \left[c_1x_1 + c_2x_2 + ... + c_px_p - (c_1\mu_1 + c_2\mu_2 + ... + c_p\mu_p) \right]^2 f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_p}(x_p) dx_1 dx_2 ... dx_p$$

$$= \int \left[c_1(x_1 - \mu_1) + ... + c_p(x_p - \mu_p) \right]^2 f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_p}(x_p) dx_1 dx_2 ... dx_p$$

Now, the cross-term

$$\int c_1 c_2 (x_1 - \mu_1)(x_2 - \mu_2) f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_p}(x_p) dx_1 dx_2 ... dx_p$$

$$= c_1 c_2 \left[\int (x_1 - \mu_1) f_{X_1}(x_1) dx_1 \right] \left[\int (x_2 - \mu_2) f_{X_2}(x_2) dx_2 \right] = 0$$

from the definition of the mean. Therefore, each cross-term in the last integral for V(Y) is zero and

$$V(Y) = \left[\int c_1^2 (x_1 - \mu_1)^2 f_{X_1}(x_1) dx_1 \right] ... \left[\int c_p^2 (x_p - \mu_p)^2 f_{X_p}(x_p) dx_p \right]$$

$$= c_1^2 V(X_1) + ... + c_p^2 V(X_p).$$

5-124 $\int_0^a \int_0^b f_{XY}(x, y)dydx = \int_0^a \int_0^b cdydx = cab$. Therefore, c = 1/ab. Then,

$f_X(x) = \int_0^b cdy = \frac{1}{a}$ for 0 < x < a, and $f_Y(y) = \int_0^a cdx = \frac{1}{b}$ for 0 < y < b. Therefore,

$f_{XY}(x,y)=f_X(x)f_Y(y)$ for all x and y and X and Y are independent.

5-125 $f_X(x) = \int_0^b g(x)h(y)dy = g(x)\int_0^b h(y)dy = kg(x)$ *where* $k = \int_0^b h(y)dy$. Also,

$f_Y(y) = lh(y)$ *where* $l = \int_0^a g(x)dx$. Because $f_{XY}(x,y)$ is a probability density

function, $\int_0^a \int_0^b g(x)h(y)dydx = \left[\int_0^a g(x)dx\right]\left[\int_0^b h(y)dy\right] = 1$. Therefore, $kl = 1$ and

$f_{XY}(x,y)=f_X(x)f_Y(y)$ for all x and y.

Section 5-8 on CD

S5-1. $f_Y(y) = \dfrac{1}{4}$ at y = 3, 5, 7, 9 from Theorem S5-1.

S5-2. Because $X \geq 0$, the transformation is one-to-one; that is $y = x^2$ and $x = \sqrt{y}$. From Theorem S5-2,

$f_Y(y) = f_X(\sqrt{y}) = \binom{3}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{3-\sqrt{y}}$ for y = 0, 1, 4, 9.

If p = 0.25, $f_Y(y) = \binom{3}{\sqrt{y}} (0.25)^{\sqrt{y}} (0.75)^{3-\sqrt{y}}$ for y = 0, 1, 4, 9.

S5-3. a) $f_Y(y) = f_X\left(\dfrac{y-10}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{y-10}{72}$ for $10 \leq y \leq 22$

b) $E(Y) = \int_{10}^{22} \dfrac{y^2 - 10y}{72} dy = \dfrac{1}{72}\left(\frac{y^3}{3} - \frac{10y^2}{2}\right)\Bigg|_{10}^{22} = 18$

S5-4. Because y = -2 ln x, $e^{-\frac{y}{2}} = x$. Then, $f_Y(y) = f_X(e^{\frac{-y}{2}})\left|-\frac{1}{2}e^{\frac{-y}{2}}\right| = \frac{1}{2}e^{\frac{-y}{2}}$ for $0 \leq e^{\frac{-y}{2}} \leq 1$ or

$y \geq 0$, which is an exponential distribution (which equals a chi-square distribution with k = 2 degrees of freedom).

S5-5. a) Let Q = R. Then,

$$p = i^2 r \qquad \text{and} \qquad i = \sqrt{\tfrac{p}{q}}$$
$$q = r \qquad\qquad\qquad r = q$$

$$J = \begin{vmatrix} \frac{\partial i}{\partial p} & \frac{\partial i}{\partial q} \\ \frac{\partial r}{\partial p} & \frac{\partial r}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(pq)^{-1/2} & -\frac{1}{2}p^{1/2}q^{-3/2} \\ 0 & 1 \end{vmatrix} = \tfrac{1}{2}(pq)^{-1/2}$$

$$f_{PQ}(p,q) = f_{IR}(\sqrt{\tfrac{p}{q}}, q)\tfrac{1}{2}(pq)^{-1/2} = 2\left(\sqrt{\tfrac{p}{q}}\right)\tfrac{1}{2}(pq)^{-1/2} = q^{-1}$$

$$\text{for } 0 \le \sqrt{\tfrac{p}{q}} \le 1,\ 0 \le q \le 1$$

That is, for $0 \le p \le q,\ 0 < q \le 1$.

$$f_P(p) = \int_p^1 q^{-1}dq = -\ln \mathrm{p} \quad \text{for } 0 < p \le 1.$$

b) $E(P) = -\int_0^1 p \ln p\ dp$. Let u = ln p and dv = p dp. Then, du = 1/p and

$v = \dfrac{p^2}{2}$. Therefore, $E(P) = -(\ln p)\frac{p^2}{2}\Big|_0^1 + \int_0^1 \frac{p}{2}dp = \frac{p^2}{4}\Big|_0^1 = \dfrac{1}{4}$

S5-6. a) If $y = x^2$, then $x = \sqrt{y}$ for $x \ge 0$ and $y \ge 0$. Thus, $f_Y(y) = f_X(\sqrt{y})\frac{1}{2}y^{-\frac{1}{2}} = \dfrac{e^{-\sqrt{y}}}{2\sqrt{y}}$ for y > 0.

b) If $y = x^{1/2}$, then $x = y^2$ for $x \ge 0$ and $y \ge 0$. Thus, $f_Y(y) = f_X(y^2)2y = 2ye^{-y^2}$ for y > 0.

c) If y = ln x, then $x = e^y$ for $x \ge 0$. Thus, $f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y} = e^{y-e^y}$ for $-\infty < y < \infty$.

S5-7. a) Now, $\int_0^\infty av^2e^{-bv}dv$ must equal one. Let u = bv, then $1 = a\int_0^\infty (\frac{u}{b})^2 e^{-u}\dfrac{du}{b} = \dfrac{a}{b^3}\int_0^\infty u^2 e^{-u}du.$ From the definition of the gamma function the last expression is $\dfrac{a}{b^3}\Gamma(3) = \dfrac{2a}{b^3}$. Therefore, $a = \dfrac{b^3}{2}$.

b) If $w = \dfrac{mv^2}{2}$, then $v = \sqrt{\dfrac{2w}{m}}$ for $v \ge 0,\ \ w \ge 0$.

$$f_W(w) = f_V\left(\sqrt{\tfrac{2w}{m}}\right)\frac{dv}{dw} = \frac{b^3 2w}{2m}e^{-b\sqrt{\frac{2w}{m}}}(2mw)^{-1/2}$$
$$= \frac{b^3 m^{-3/2}}{\sqrt{2}}w^{1/2}e^{-b\sqrt{\frac{2w}{m}}}$$

for $w \ge 0$.

S5-8. If $y = e^x$, then x = ln y for $1 \le x \le 2$ and $e^1 \le y \le e^2$. Thus, $f_Y(y) = f_X(\ln y)\frac{1}{y} = \frac{1}{y}$ for $1 \le \ln y \le 2$. That is, $f_Y(y) = \frac{1}{y}$ for $e \le y \le e^2$.

S5-9. Now $P(Y \le a) = P(X \ge u(a)) = \int_{u(a)}^{\infty} f_X(x)dx$. By changing the variable of integration from x to y by using x = u(y), we obtain $P(Y \le a) = \int_{a}^{-\infty} f_X(u(y))u'(y)dy$ because as x tends to ∞, y = h(x) tends to - ∞. Then, $P(Y \le a) = \int_{-\infty}^{a} f_X(u(y))(-u'(y))dy$. Because h(x) is decreasing, $u'(y)$ is negative. Therefore, $|u'(y)| = -u'(y)$ and Theorem S5-1 holds in this case also.

S5-10. If y = $(x-2)^2$, then x = $2-\sqrt{y}$ for $0 \le x \le 2$ and x = $2+\sqrt{y}$ for $2 \le x \le 4$. Thus,

$$
\begin{aligned}
f_Y(y) &= f_X(2-\sqrt{y})\,|-\tfrac{1}{2}y^{-1/2}\,| + f_X(2+\sqrt{y})\,|\tfrac{1}{2}y^{-1/2}\,| \\
&= \frac{2-\sqrt{y}}{16\sqrt{y}} + \frac{2+\sqrt{y}}{16\sqrt{y}} \\
&= \left(\tfrac{1}{4}\right)y^{-1/2} \text{ for } 0 \le y \le 4
\end{aligned}
$$

S5-11. a) Let $a = s_1 s_2$ and y = s_1. Then, s_1 = y, $s_2 = \frac{a}{y}$ and

$$
J = \begin{vmatrix} \frac{\partial s_1}{\partial a} & \frac{\partial s_1}{\partial y} \\ \frac{\partial s_2}{\partial a} & \frac{\partial s_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{y} & -\frac{a}{2}y^{-3/2} \end{vmatrix} = -\frac{1}{y}. \text{ Then,}
$$

$f_{AY}(a, y) = f_{S_1S_2}(y, \tfrac{a}{y})(\tfrac{1}{y}) = 2y(\tfrac{a}{8y})(\tfrac{1}{y}) = \tfrac{a}{4y}$ for $0 \le y \le 1$ and $0 \le \frac{a}{y} \le 4$. That is, for $0 \le y \le 1$ and $0 \le a \le 4y$.

b) $f_A(a) = \int_{a/4}^{1} \frac{a}{4y}dy = -\frac{a}{4}\ln(\tfrac{a}{4})$ for $0 < a \le 4$.

S5-12. Let r = v/i and s = i. Then, i = s and v = rs

$$J = \begin{vmatrix} \frac{\partial i}{\partial r} & \frac{\partial i}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ s & r \end{vmatrix} = s$$

$f_{RS}(r,s) = f_{IV}(s, rs)s = e^{-rs}\text{s}$ for $\text{rs} \geq 0$ and $1 \leq s \leq 2$. That is,

$f_{RS}(r,s) = se^{-rs}$ for $1 \leq \text{s} \leq 2$ and $\text{r} \geq 0$.

Then, $f_R(r) = \int_1^2 se^{-rs}ds$. Let u = s and dv = $\text{e}^{-\text{rs}}$ ds. Then, du = ds and $v = \dfrac{-e^{-rs}}{r}$

Then,

$$f_R(r) = -s\frac{-e^{-rs}}{r}\bigg|_1^2 + \int_1^2 \frac{-e^{-rs}}{r}ds = \frac{e^{-r} - 2e^{-2r}}{r} - \frac{e^{-rs}}{r^2}\bigg|_1^2$$

$$= \frac{e^{-r} - 2e^{-2r}}{r} + \frac{e^{-r} - e^{-2r}}{r^2}$$

$$= \frac{e^{-r}(r+1) - e^{-2r}(2r+1)}{r^2}$$

for r > 0.

Section 5-9 on CD

S5-13 . a) $E(e^{tX}) = \sum_{x=1}^{m} \frac{e^{tx}}{m} = \frac{1}{m}\sum_{x=1}^{m}(e^t)^x = \frac{(e^t)^{m+1} - e^t}{m(e^t - 1)} = \frac{e^t(1 - e^{tm})}{m(1 - e^t)}$

b) $M(t) = \frac{1}{m}e^t(1 - e^{tm})(1 - e^t)^{-1}$ and

$$\frac{dM(t)}{dt} = \frac{1}{m}\left\{e^t(1 - e^{tm})(1 - e^t)^{-1} + e^t(-me^{tm})(1 - e^t)^{-1} + e^t(1 - e^{tm})(-1)(1 - e^t)^{-2}(-e^t)\right\}$$

$$\frac{dM(t)}{dt} = \frac{e^t}{m(1 - e^t)}\left\{1 - e^{tm} - me^{tm} + \frac{(1 - e^{tm})e^t}{1 - e^t}\right\}$$

$$= \frac{e^t}{m(1 - e^t)^2}\left\{1 - e^{tm} - me^{tm} + me^{(m+1)t}\right\}$$

Using L'Hospital's rule,

$$\lim_{t\to 0}\frac{dM(t)}{dt} = \lim_{t\to 0}\frac{e^t}{m}\lim_{t\to 0}\frac{-me^{tm} - m^2e^{tm} + m(m+1)e^{(m+1)t}}{-2(1 - e^t)e^t}$$

$$= \lim_{t\to 0}\frac{e^t}{m}\lim_{t\to 0}\frac{-m^2e^{tm} - m^3e^{tm} + m(m+1)^2e^{(m+1)t}}{-2(1 - e^t)e^t - 2e^t(-e^t)}$$

$$= \frac{1}{m} \times \frac{m(m+1)^2 - m^2 - m^3}{2} = \frac{m^2 + m}{2m} = \frac{m+1}{2}$$

Therefore, E(X) = $\frac{m+1}{2}$.

$$\frac{d^2M(t)}{dt^2} = \frac{d^2}{dt^2}\sum_{x=1}^{m} e^{tx}\frac{1}{m} = \frac{1}{m}\sum_{x=1}^{m}\frac{d^2}{dt^2}\left(1 + tx + \frac{(tx)^2}{2} + \ldots\right)$$

$$= \frac{1}{m}\sum_{x=1}^{m}(x^2 + term\sin volvingpow ersoft)$$

Thus,

$$\left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = \frac{1}{m}\left(\frac{m(m+1)(m+2)}{6}\right) = \frac{(m+1)(2m+2)}{6}$$

Then,

$$V(X) = \frac{2m^2 + 3m + 1}{6} - \frac{(m+1)^2}{4} = \frac{4m^2 + 6m + 2 - 3m^2 - 6m - 3}{12}$$

$$= \frac{m^2 - 1}{12}$$

S5-14. a) $E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$

b) $\frac{dM(t)}{dt} = \lambda e^t e^{\lambda(e^t-1)}$

$$\left.\frac{dM(t)}{dt}\right|_{t=0} = \lambda = E(X)$$

$$\frac{d^2M(t)}{dt^2} = \lambda^2 e^{2t} e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}$$

$$\left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = \lambda^2 + \lambda$$

$$V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

S5-15 a) $E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p = \frac{p}{1-p}\sum_{x=1}^{\infty}[e^t(1-p)]^x$

$$= \frac{e^t(1-p)}{1-(1-p)e^t}\left(\frac{p}{1-p}\right) = \frac{pe^t}{1-(1-p)e^t}$$

b) $\frac{dM(t)}{dt} = pe^t(1-(1-p)e^t)^{-2}(1-p)e^t + pe^t(1-(1-p)e^t)^{-1}$

$$= p(1-p)e^{2t}(1-(1-p)e^t)^{-2} + pe^t(1-(1-p)e^t)^{-1}$$

$$\left.\frac{dM(t)}{dt}\right|_{t=0} = \frac{1-p}{p} + 1 = \frac{1}{p} = E(X)$$

$$\frac{d^2M(t)}{dt^2} = p(1-p)e^{2t}2(1-(1-p)e^t)^{-3}(1-p)e^t + p(1-p)(1-(1-p)e^t)^{-2}2e^{2t}$$

$$+ pe^t(1-(1-p)e^t)^{-2}(1-p)e^t + pe^t(1-(1-p)e^t)^{-1}$$

$$\left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = \frac{2(1-p)^2}{p^2} + \frac{2(1-p)}{p} + \frac{1-p}{p} + 1 = \frac{2(1-p)^2 + 3p(1-p) + p^2}{p^2}$$

$$= \frac{2 + 2p^2 - 4p + 3p - 3p^2 + p^2}{p^2} = \frac{2-p}{p^2}$$

$$V(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

S5-16. $M_Y(t) = Ee^{tY} = Ee^{t(X_1+X_2)} = Ee^{tX_1}Ee^{tX_2}$

$$= (1-2t)^{-k_1/2}(1-2t)^{-k_2/2} = (1-2t)^{-(k_1+k_2)/2}$$

Therefore, Y has a chi-square distribution with $k_1 + k_2$ degrees of freedom.

S5-17. a) $E(e^{tX}) = \int_0^{\infty} e^{tx}4xe^{-2x}dx = 4\int_0^{\infty} xe^{(t-2)x}dx$

Using integration by parts with *u* = x and *dv* = $e^{(t-2)x}dx$ and *du* = dx,

v = $\frac{e^{(t-2)x}}{t-2}$ we obtain

$$4\left(\left.\frac{xe^{(t-2)x}}{t-2}\right|_0^{\infty} - \int_0^{\infty}\frac{e^{(t-2)x}}{t-2}dx\right) = 4\left(\left.\frac{xe^{(t-2)x}}{t-2}\right|_0^{\infty} - \left.\frac{e^{(t-2)x}}{(t-2)^2}\right|_0^{\infty}\right)$$

This integral only exists for t < 2. In that case, $E(e^{tX}) = \frac{4}{(t-2)^2}$ for t < 2

b) $\frac{dM(t)}{dt} = -8(t-2)^{-3}$ and $\left.\frac{dM(t)}{dt}\right|_{t=0} = -8(-2)^{-3} = 1 = E(X)$

c) $\frac{d^2M(t)}{dt^2} = 24(t-2)^{-4}$ and $\left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = \frac{24}{16} = \frac{3}{2}$. Therefore, $V(X) = \frac{3}{2} - 1^2 = \frac{1}{2}$

S5-18. a) $E(e^{tX}) = \int_{\alpha}^{\beta} \frac{e^{tx}}{\beta - \alpha} dx = \left. \frac{e^{tx}}{t(\beta - \alpha)} \right|_{\alpha}^{\beta} = \frac{e^{t\beta} - e^{t\alpha}}{t(\beta - \alpha)}$

b) $\frac{dM(t)}{dt} = \frac{e^{t\beta} - e^{t\alpha}}{-(\beta - \alpha)t^2} + \frac{\beta e^{t\beta} - \alpha e^{t\alpha}}{t(\beta - \alpha)}$

$$= \frac{(\beta t - 1)e^{t\beta} - (\alpha t - 1)e^{t\alpha}}{t^2(\beta - \alpha)}$$

Using L'Hospital's rule,

$$\lim_{t \to 0} \frac{dM(t)}{dt} = \frac{(\beta t - 1)\beta e^{t\beta} + \beta e^{t\beta} - (\alpha t - 1)\alpha e^{t\alpha} - \alpha e^{t\alpha}}{2t(\beta - \alpha)}$$

$$\lim_{t \to 0} \frac{dM(t)}{dt} = \frac{\beta^2(\beta t - 1)e^{t\beta} + \beta^2 e^{t\beta} + \beta^2 e^{t\beta} - \alpha^2(\alpha t - 1)e^{t\alpha} - \alpha^2 e^{t\alpha} - \alpha^2 e^{t\alpha}}{2(\beta - \alpha)}$$

$$\lim_{t \to 0} \frac{dM(t)}{dt} = \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{(\beta + \alpha)}{2} = E(X)$$

$$\frac{d^2 M(t)}{dt^2} = \frac{d^2}{dt^2} \int_a^b \frac{1}{b-a} e^{tx} dx = \frac{1}{b-a} \frac{d^2}{dt^2} \left(\frac{e^{tb} - e^{ta}}{t} \right)$$

$$= \frac{1}{b-a} \frac{d^2}{dt^2} \left(\frac{tb + \frac{(tb)^2}{2} + \frac{(tb)^3}{3!} - ta - \frac{(ta)^2}{2} - \frac{(ta)^3}{3!} + \ldots}{t} \right) =$$

$$= \frac{1}{b-a} \frac{d^2}{dt^2} \left(b - a + \frac{(b^2 - a^2)t}{2} + \frac{(b^3 - a^3)t^2}{3!} + \ldots \right)$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ba + a^2}{3}$$

Thus,

$$V(X) = \frac{b^2 + ba + a^2}{3} - \frac{(b+a)^2}{4} = \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

S5-19. a) $M(t)=\int_0^\infty e^{tx}\lambda e^{-\lambda x}dx=\lambda\int_0^\infty e^{(t-\lambda)x}dx$

$$=\lambda\frac{e^{(t-\lambda)x}}{t-\lambda}\bigg|_0^\infty=\frac{-\lambda}{t-\lambda}=\frac{1}{1-\frac{t}{\lambda}}=\left(1-\tfrac{t}{\lambda}\right)^{-1} \text{ for t < λ}$$

b) $\dfrac{dM(t)}{dt}=(-1)\left(1-\tfrac{t}{\lambda}\right)^{-2}\left(\tfrac{-1}{\lambda}\right)=\dfrac{1}{\lambda\left(1-\frac{t}{\lambda}\right)^2}$

$$\frac{dM(t)}{dt}\bigg|_{t=0}=\frac{1}{\lambda}$$

$$\frac{d^2M(t)}{dt^2}=\frac{2}{\lambda^2(1-\frac{t}{\lambda})^3}$$

$$\frac{d^2M(t)}{dt^2}\bigg|_{t=0}=\frac{2}{\lambda^2}$$

$$V(X)=\frac{2}{\lambda^2}-\left(\frac{1}{\lambda}\right)^2=\frac{1}{\lambda^2}$$

S5-20. a) $M(t)=\int_0^\infty e^{tx}\dfrac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}dx=\dfrac{\lambda^r}{\Gamma(r)}\int_0^\infty x^{r-1}e^{(t-\lambda)x}dx$

Let u = (λ-t)x. Then,

$$M(t)=\frac{\lambda^r}{\Gamma(r)}\int_0^\infty\left(\frac{u}{\lambda-t}\right)^{r-1}e^{-u}\frac{du}{\lambda-t}=\frac{\lambda^r\Gamma(r)}{\Gamma(r)(\lambda-t)^r}=\frac{1}{(1-\frac{t}{\lambda})^r}=(1-\tfrac{t}{\lambda})^{-r} \text{ from}$$

the definition of the gamma function for t < λ.

b) $M'(t)=-r(1-\frac{t}{\lambda})^{-r-1}(-\frac{1}{\lambda})$

$$M'(t)\big|_{t=0}=\frac{r}{\lambda}=E(X)$$

$$M''(t)=\frac{r(r+1)}{\lambda^2}(1-\tfrac{t}{\lambda})^{r-2}$$

$$M''(t)\big|_{t=0}=\frac{r(r+1)}{\lambda^2}$$

$$V(X)=\frac{r(r+1)}{\lambda^2}-\left(\frac{r}{\lambda}\right)^2=\frac{r}{\lambda^2}$$

S5-21. a) $E(e^{tY}) = \prod_{i=1}^{n} E(e^{tX_i}) = \left(1 - \frac{t}{\lambda}\right)^{-n}$

b) From Exercise S5-20, Y has a gamma distribution with parameter λ and n.

S5-22. a) $M_Y(t) = e^{\mu_1 t + \sigma_1^2 \frac{t^2}{2} + \mu_2 t + \sigma_2^2 \frac{t^2}{2}} = e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)\frac{t^2}{2}}$

b) Y has a normal distribution with mean $\mu_1 + \mu_2$ and variances $\sigma_1^2 + \sigma_2^2$

S5-23. Because a chi-square distribution is a special case of the gamma distribution with $\lambda = \frac{1}{2}$ and $r = \frac{k}{2}$, from Exercise S5-20.

$$M(t) = (1-2t)^{-k/2}$$

$$M'(t) = -\frac{k}{2}(1-2t)^{-\frac{k}{2}-1}(-2) = k(1-2t)^{-\frac{k}{2}-1}$$

$$M'(t)\Big|_{t=0} = k = E(X)$$

$$M''(t) = 2k(\tfrac{k}{2}+1)(1-2t)^{-\frac{k}{2}-2}$$

$$M''(t)\Big|_{t=0} = 2k(\tfrac{k}{2}+1) = k^2 + 2k$$

$$V(X) = k^2 + 2k - k^2 = 2k$$

S5-24. a) $M(t) = M(0) + M'(0)t + M'(0)\frac{t^2}{2!} + \ldots + M^{(r)}(0)\frac{t^r}{r!} + \ldots$ by Taylor's expansion. Now, M(0) = 1 and $M^{(r)}(0) = \mu_r'$ and the result is obtained.

b) From Exercise S5-20, $M(t) = 1 + \frac{r}{\lambda}t + \frac{r(r+1)}{\lambda^2}\frac{t^2}{2!} + \ldots$

c) $\mu_1' = \frac{r}{\lambda}$ and $\mu_2' = \frac{r(r+1)}{\lambda^2}$ which agrees with Exercise S5-20.

Section 5-10 on CD

S5-25. Use Chebychev's inequality with c = 4. Then, $P(|X - 10| > 4) \le \frac{1}{16}$.

S5-26. E(X) = 5 and $\sigma_X = 2.887$. Then, $P(|X - 5| > 2\sigma_X) \le \frac{1}{4}$.

The actual probability is $P(|X - 5| > 2\sigma_X) = P(|X - 5| > 5.77) = 0$.

S5-27. E(X) = 20 and V(X) = 400. Then, $P(|X-20|>2\sigma)\le\frac{1}{4}$ and $P(|X-20|>3\sigma)\le\frac{1}{9}$ The actual probabilities are

$$P(|X-20|>2\sigma)=1-P(|X-20|<40)$$
$$=1-\int_0^{60}0.05e^{-0.05x}dx=1-\left[-e^{-0.05x}\Big|_0^{60}\right]=0.0498$$
$$P(|X-20|>3\sigma)=1-P(|X-20|<60)$$
$$=1-\int_0^{80}0.05e^{-0.05x}dx=1-\left[-e^{-0.05x}\Big|_0^{80}\right]=0.0183$$

S5-28. E(X) = 4 and $\sigma_X = 2$

$P(|X-4|\ge 4)\le\frac{1}{4}$ *and* $P(|X-4|\ge 6)\le\frac{1}{9}$. The actual probabilities are

$$P(|X-4|\ge 4)=1-P(|X-4|<4)=1-\sum_{x=1}^{7}\tfrac{e^{-2}2^x}{x!}=1-0.8636=0.1364$$
$$P(|X-4|\ge 6)=1-P(|X-4|<6)=1-\sum_{x=1}^{9}\tfrac{e^{-2}2^x}{x!}=0.000046$$

S5-29. Let $\overline{X}$ denote the average of 500 diameters. Then, $\sigma_{\overline{X}}=\frac{0.01}{\sqrt{500}}=4.47x10^{-4}$.

a) $P(|\overline{X}-\mu|\ge 4\sigma_{\overline{X}})\le\frac{1}{16}$ *and* $P(|\overline{X}-\mu|<0.0018)\ge\frac{15}{16}$. Therefore, the bound is 0.0018.

If $P(|\overline{X}-\mu|<x)=\frac{15}{16}$, then $P(\frac{-x}{\sigma_{\overline{X}}}<\frac{\overline{X}-\mu}{\sigma_{\overline{X}}}<\frac{x}{\sigma_{\overline{X}}})=0.9375.$ Then,

$P(\frac{-x}{4.47x10^{--4}}<Z<\frac{x}{4.47x10^{-4}})=0.9375.$ and $\frac{x}{4.47x10^{-4}}$ = 1.86. Therefore, x = $8.31x10^{-4}$.

S5-30. a) E(Y) = $P(|X-\mu|\ge c\sigma)$

b) Because $Y\le 1$, $(X-\mu)^2\ge(X-\mu)^2Y$

If $|X-\mu|\ge c\sigma$, then Y = 1 and $(X-\mu)^2Y\ge c^2\sigma^2Y$

If $|X-\mu|<c\sigma$, then Y = 0 and $(X-\mu)^2Y=c^2\sigma^2Y$.

c) Because $(X-\mu)^2\ge c^2\sigma^2Y$, $E[(X-\mu)^2]\ge c^2\sigma^2E(Y)$.

d) From part a., E(Y) = $P(|X-\mu|\ge c\sigma)$. From part c., $\sigma^2\ge c^2\sigma^2P(|X-\mu|\ge c\sigma)$. Therefore,

$\frac{1}{c^2}\ge P(|X-\mu|\ge c\sigma)$.